

HYDROGEN ATOM: RADIAL FUNCTIONS FOR LARGE L

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.46.

In solving the radial equation for the hydrogen atom, we arrived at the solution

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho) \quad (1)$$

$$u(r) \equiv rR(r) \quad (2)$$

$$\rho = \kappa r \quad (3)$$

$$\rho_0 = \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa} \quad (4)$$

$$\kappa = \frac{\sqrt{-2mE}}{\hbar} \quad (5)$$

with $v(\rho)$ given by a series

$$v(\rho) = \sum_{j=0}^{\infty} c_j \rho^j \quad (6)$$

The coefficients turned out to satisfy the recursion formula

$$c_{j+1} = \frac{2(j+l+1) - 2n}{(j+1)(j+2(l+1))} c_j \quad (7)$$

For $l = n - 1$, the recursion formula becomes

$$c_{j+1} = \frac{2j}{(j+1)(j+2n)} c_j \quad (8)$$

where we need to specify c_0 to get the series started. Working back to the radial function gives us

$$R_{n(n-1)}(r) = \frac{c_0}{(an)^n} r^{n-1} e^{-r/an} \quad (9)$$

We can combine the constant out front into a single constant $N_n = c_0/(an)^n$.

To normalize, we evaluate the integral (using software)

$$\int_0^{\infty} R_{n(n-1)}^2(r) dr = \frac{c_0^2}{(an)^{2n}} \int_0^{\infty} r^{2(n-1)} e^{-2r/an} r^2 dr \quad (10)$$

$$= \frac{na(2n)!}{2^{2n+1}} c_0^2 \quad (11)$$

$$= 1 \quad (12)$$

$$c_0 = 2^n \sqrt{\frac{2}{an(2n)!}} \quad (13)$$

$$N_n = 2^n \sqrt{\frac{2}{(an)^{2n+1}(2n)!}} \quad (14)$$

Expectation values are calculated in the usual way:

$$\langle r \rangle = \frac{2^{2n+1}}{(an)^{2n+1}(2n)!} \int_0^{\infty} r^{2(n-1)+3} e^{-2r/an} dr \quad (15)$$

$$= \frac{(2n+1)na}{2} \quad (16)$$

$$\langle r^2 \rangle = \frac{2^{2n+1}}{(an)^{2n+1}(2n)!} \int_0^{\infty} r^{2(n-1)+4} e^{-2r/an} dr \quad (17)$$

$$= \frac{(n+1)(2n+1)n^2 a^2}{2} \quad (18)$$

$$\sigma_r = \sqrt{\langle r^2 \rangle - \langle r \rangle^2} \quad (19)$$

$$= \frac{na\sqrt{2n+1}}{2} \quad (20)$$

$$= \frac{\langle r \rangle}{\sqrt{2n+1}} \quad (21)$$

A few radial functions $R_{n(n-1)}(r)$ are shown. The domain of r is up to $200a$ in each case.





