HYDROGEN ATOM: RADIAL FUNCTIONS FOR LARGE L

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.46.

In solving the radial equation for the hydrogen atom, we arrived at the solution

$$\begin{aligned}
u(\rho) &= \rho^{l+1}e^{-\rho}v(\rho) && (1)\\
u(r) &\equiv rR(r) && (2)\\
\rho &= \kappa r && (3)\\
\rho_0 &= \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa} && (4)\\
\kappa &= \frac{\sqrt{-2mE}}{\hbar} && (5)
\end{aligned}$$

with $v(\rho)$ given by a series

$$v(\rho)=\sum_{j=0}^{\infty}c_j\rho^j \tag{6}$$

The coefficients turned out to satisfy the recursion formula

$$c_{j+1}=\frac{2(j+l+1)-2n}{(j+1)(j+2(l+1))}c_j \tag{7}$$

For $l=n-1$, the recursion formula becomes

$$c_{j+1}=\frac{2j}{(j+1)(j+2n)}c_j \tag{8}$$

where we need to specify c_0 to get the series started. Working back to the radial function gives us

$$R_{n(n-1)}(r)=\frac{c_0}{(an)^n}r^{n-1}e^{-r/an} \tag{9}$$

We can combine the constant out front into a single constant $N_n=c_0/(an)^n$.

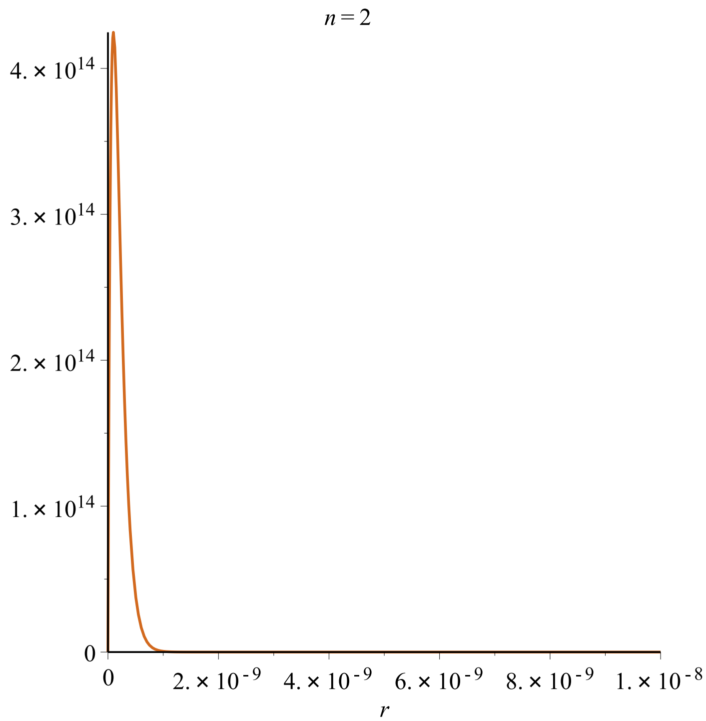
To normalize, we evaluate the integral (using software)

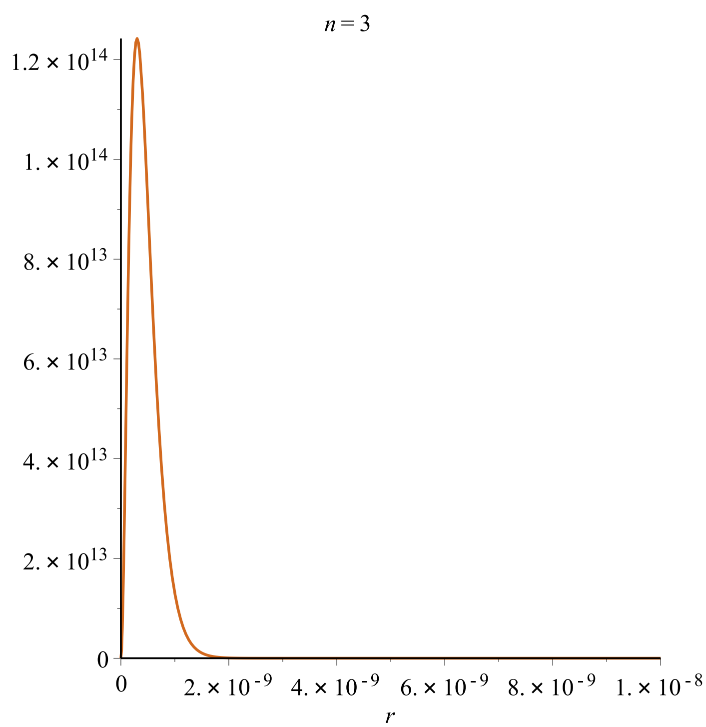
$$
\begin{aligned}
\int_0^\infty R_{n(n-1)}^2(r)dr &= \frac{c_0^2}{(an)^{2n}}\int_0^\infty r^{2(n-1)}e^{-2r/an}r^2dr && (10)\\
&= \frac{na(2n)!}{2^{2n+1}}c_0^2 && (11)\\
&= 1 && (12)\\
c_0 &= 2^n\sqrt{\frac{2}{an(2n)!}} && (13)\\
N_n &= 2^n\sqrt{\frac{2}{(an)^{2n+1}(2n)!}} && (14)
\end{aligned}
$$

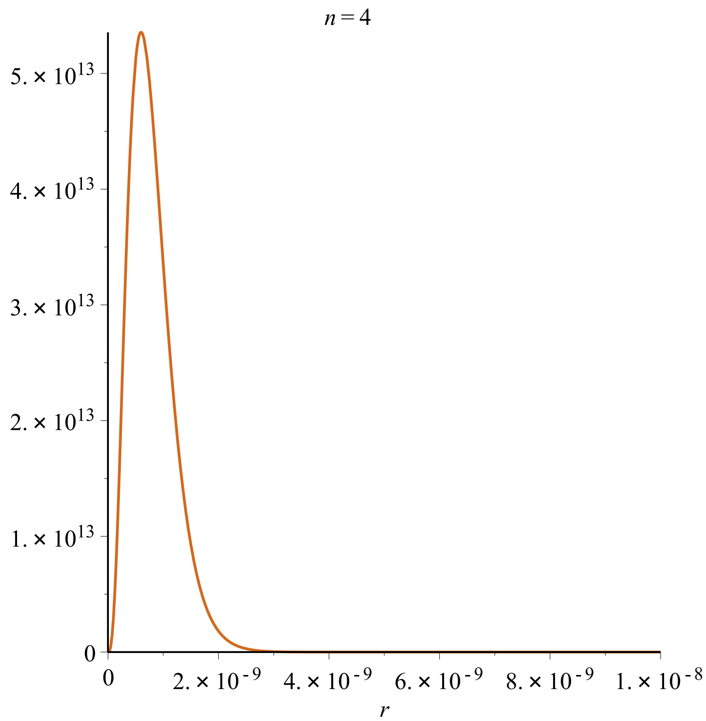
Expectation values are calculated in the usual way:

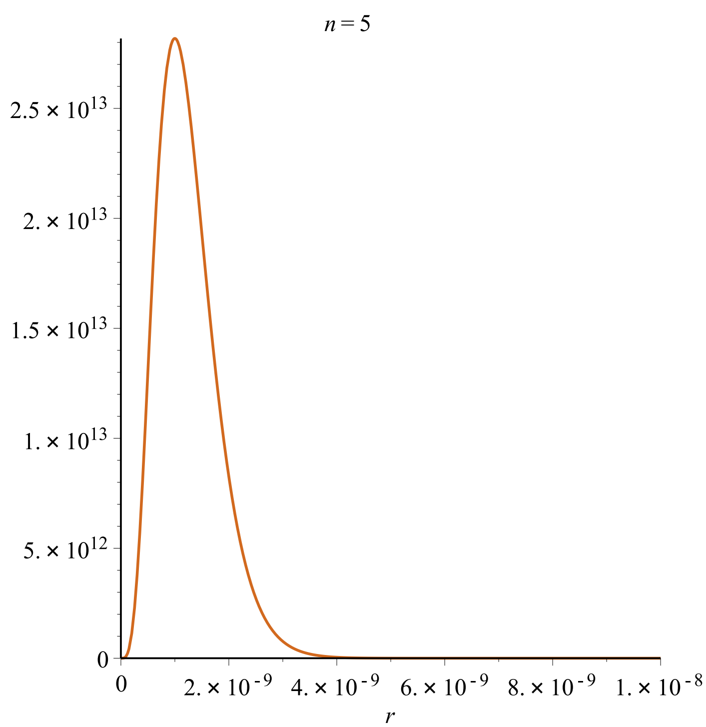
$$
\begin{aligned}
\langle r\rangle &= \frac{2^{2n+1}}{(an)^{2n+1}(2n)!}\int_0^\infty r^{2(n-1)+3}e^{-2r/an}dr && (15)\\
&= \frac{(2n+1)na}{2} && (16)\\
\langle r^2\rangle &= \frac{2^{2n+1}}{(an)^{2n+1}(2n)!}\int_0^\infty r^{2(n-1)+4}e^{-2r/an}dr && (17)\\
&= \frac{(n+1)(2n+1)n^2a^2}{2} && (18)\\
\sigma_r &= \sqrt{\langle r^2\rangle-\langle r\rangle^2} && (19)\\
&= \frac{na\sqrt{2n+1}}{2} && (20)\\
&= \frac{\langle r\rangle}{\sqrt{2n+1}} && (21)
\end{aligned}
$$

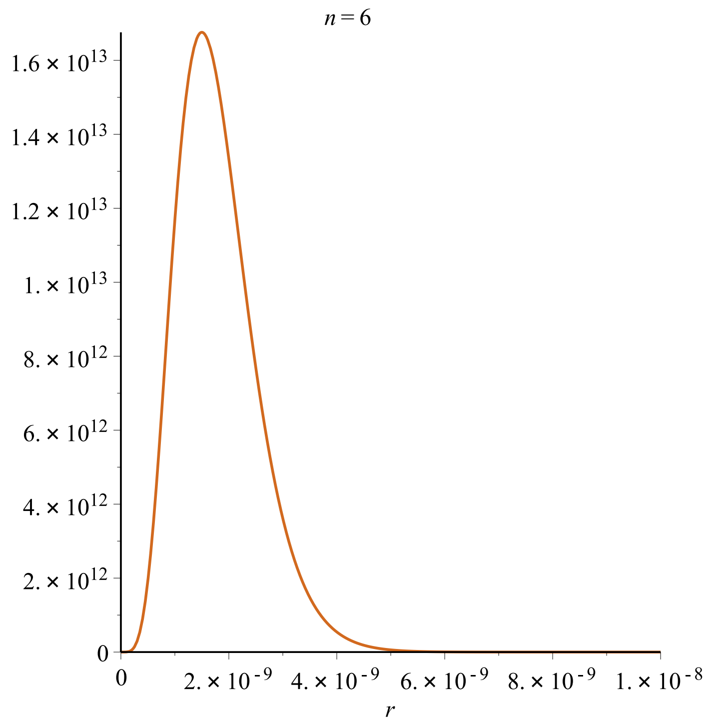
A few radial functions $R_{n(n-1)}(r)$ are shown. The domain of r is up to $200a$ in each case.

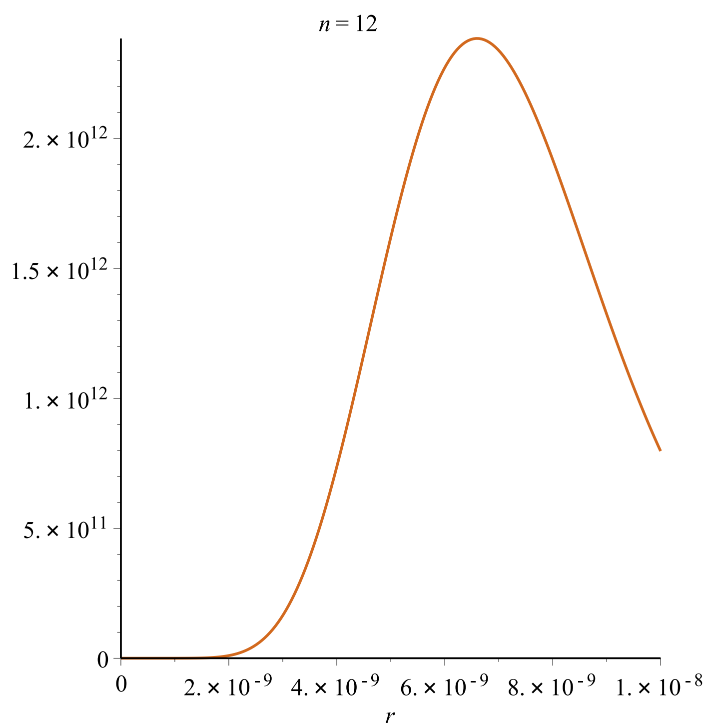

n = 2
4. × 10^14
3. × 10^14
2. × 10^14
1. × 10^14
0
0
2. × 10^-9
4. × 10^-9
6. × 10^-9
8. × 10^-9
1. × 10^-8
r


n = 3
1.2 × 10^14
1. × 10^14
8. × 10^13
6. × 10^13
4. × 10^13
2. × 10^13
0
0
2. × 10^-9
4. × 10^-9
6. × 10^-9
8. × 10^-9
1. × 10^-8
r


n = 4
5. × 10^13
4. × 10^13
3. × 10^13
2. × 10^13
1. × 10^13
0
0
2. × 10^-9
4. × 10^-9
6. × 10^-9
8. × 10^-9
1. × 10^-8
r


n = 5
2.5 × 10^13
2. × 10^13
1.5 × 10^13
1. × 10^13
5. × 10^12
0
0
2. × 10^-9
4. × 10^-9
6. × 10^-9
8. × 10^-9
1. × 10^-8
r


n = 6
1.6 × 10^13
1.4 × 10^13
1.2 × 10^13
1. × 10^13
8. × 10^12
6. × 10^12
4. × 10^12
2. × 10^12
0
0
2. × 10^-9
4. × 10^-9
6. × 10^-9
8. × 10^-9
1. × 10^-8
r


n = 12
2. × 10^12
1.5 × 10^12
1. × 10^12
5. × 10^11
0
0
2. × 10^-9
4. × 10^-9
6. × 10^-9
8. × 10^-9
1. × 10^-8
r