## HYDROGEN ATOM: RADIAL FUNCTIONS FOR LARGE L

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.46.

In solving the radial equation for the hydrogen atom, we arrived at the solution

$$u(\rho) = \rho^{l+1}e^{-\rho}v(\rho) \tag{0.1}$$

$$u(r) \equiv rR(r) \tag{0.2}$$

$$\rho = \kappa r \tag{0.3}$$

$$\rho_0 = \frac{me^2}{2\pi\varepsilon_0\hbar^2\kappa} \tag{0.4}$$

$$\kappa = \frac{\sqrt{-2mE}}{\hbar} \tag{0.5}$$

with $v(\rho)$ given by a series

$$v(\rho) = \sum_{j=0}^{\infty} c_j\rho^j \tag{0.6}$$

The coefficients turned out to satisfy the recursion formula

$$c_{j+1} = \frac{2(j+l+1)-2n}{(j+1)(j+2(l+1))}c_j \tag{0.7}$$

For $l = n-1$, the recursion formula becomes

$$c_{j+1} = \frac{2j}{(j+1)(j+2n)}c_j \tag{0.8}$$

where we need to specify $c_0$ to get the series started. Working back to the radial function gives us

$$R_{n(n-1)}(r) = \frac{c_0}{(an)^n}r^{n-1}e^{-r/an} \tag{0.9}$$

We can combine the constant out front into a single constant $N_n = c_0/(an)^n$.
To normalize, we evaluate the integral (using software)

$$\int_0^\infty R_{n(n-1)}^2(r)dr = \frac{c_0^2}{(an)^{2n}}\int_0^\infty r^{2(n-1)}e^{-2r/an}r^2dr \tag{0.10}$$

$$= \frac{na(2n)!}{2^{2n+1}}c_0^2 \tag{0.11}$$

$$= 1 \tag{0.12}$$

$$c_0 = 2^n\sqrt{\frac{2}{an(2n)!}} \tag{0.13}$$

$$N_n = 2^n\sqrt{\frac{2}{(an)^{2n+1}(2n)!}} \tag{0.14}$$

Expectation values are calculated in the usual way:

$$\langle r\rangle = \frac{2^{2n+1}}{(an)^{2n+1}(2n)!}\int_0^\infty r^{2(n-1)+3}e^{-2r/an}dr \tag{0.15}$$

$$= \frac{(2n+1)na}{2} \tag{0.16}$$

$$\langle r^2\rangle = \frac{2^{2n+1}}{(an)^{2n+1}(2n)!}\int_0^\infty r^{2(n-1)+4}e^{-2r/an}dr \tag{0.17}$$

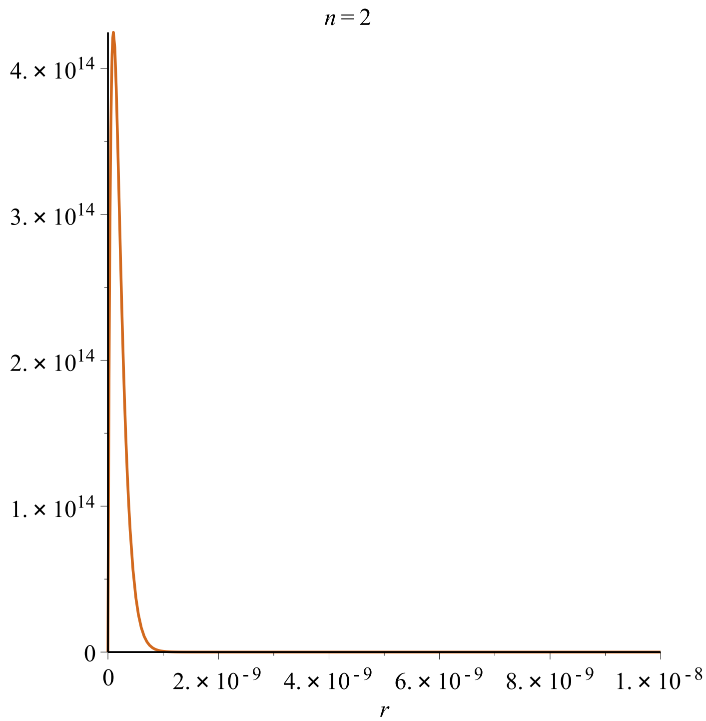
$$= \frac{(n+1)(2n+1)n^2a^2}{2} \tag{0.18}$$

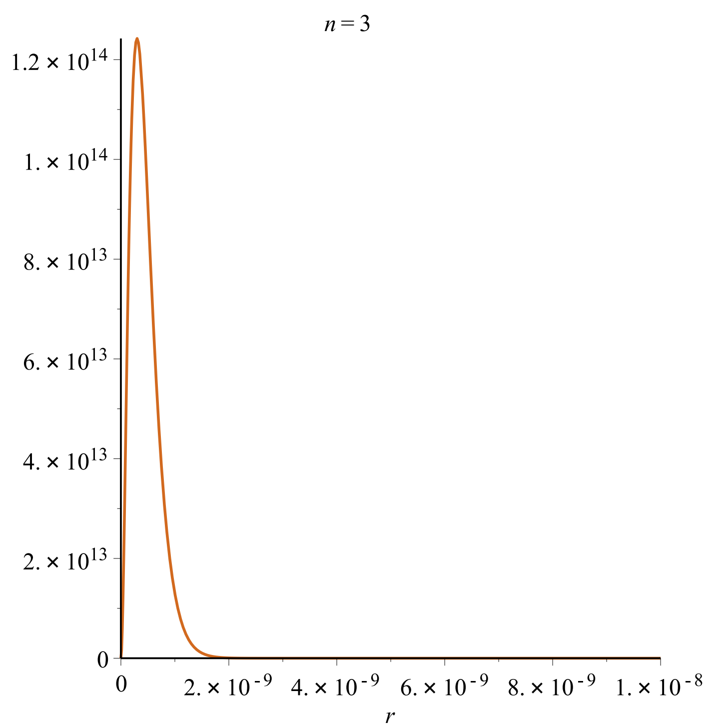
$$\sigma_r = \sqrt{\langle r^2\rangle - \langle r\rangle^2} \tag{0.19}$$

$$= \frac{na\sqrt{2n+1}}{2} \tag{0.20}$$

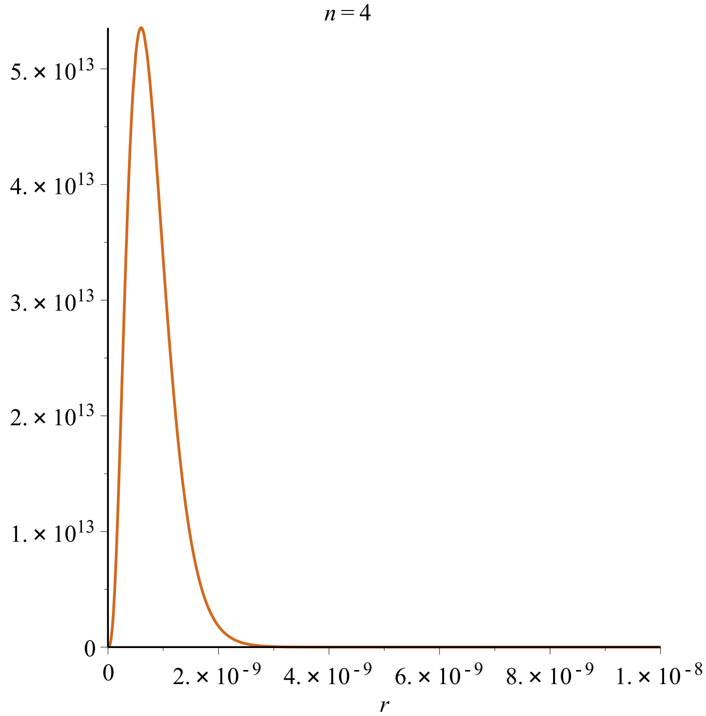
$$= \frac{\langle r\rangle}{\sqrt{2n+1}} \tag{0.21}$$

A few radial functions $R_{n(n-1)}(r)$ are shown. The domain of $r$ is up to $200a$ in each case.

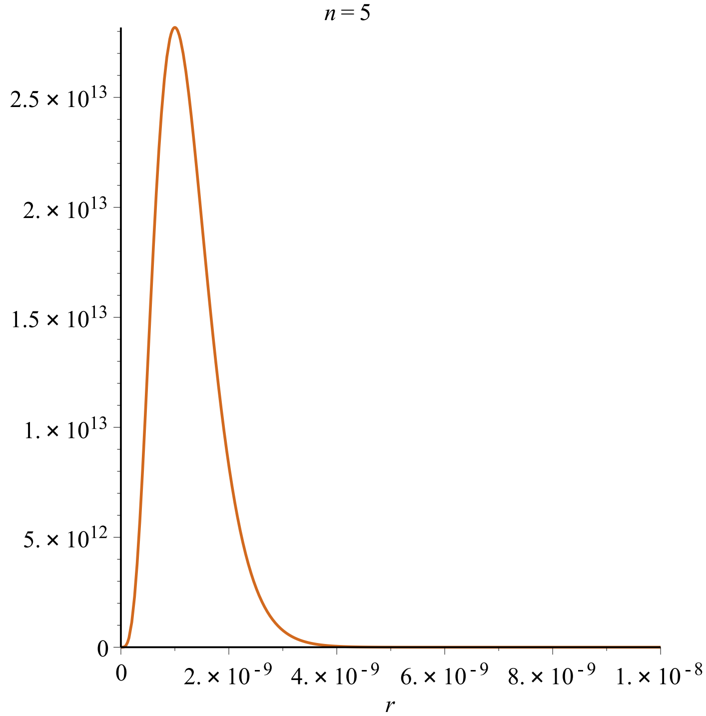

n = 2
4. × 10^14
3. × 10^14
2. × 10^14
1. × 10^14
0
0
2. × 10^-9
4. × 10^-9
6. × 10^-9
8. × 10^-9
1. × 10^-8
r

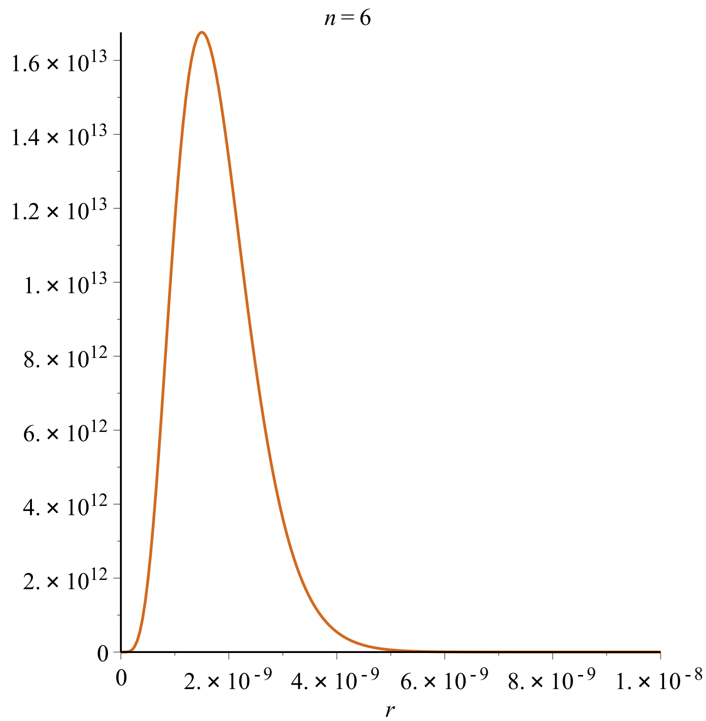

n = 3
1.2 × 10^14
1. × 10^14
8. × 10^13
6. × 10^13
4. × 10^13
2. × 10^13
0
0
2. × 10^-9
4. × 10^-9
6. × 10^-9
8. × 10^-9
1. × 10^-8
r

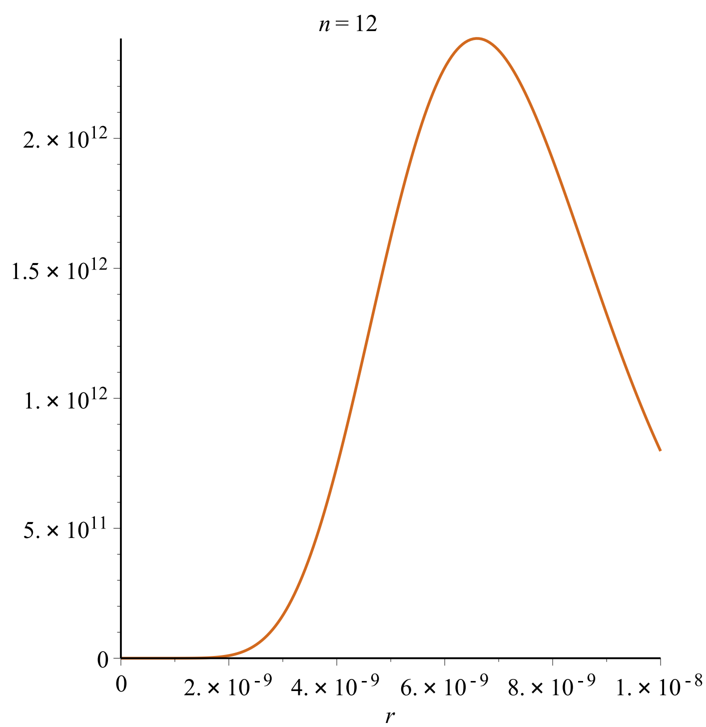
$n = 4$


5. × 10^13
4. × 10^13
3. × 10^13
2. × 10^13
1. × 10^13
0
0
2. × 10^-9
4. × 10^-9
6. × 10^-9
8. × 10^-9
1. × 10^-8
r

$n = 5$


2.5 × 10^13
2. × 10^13
1.5 × 10^13
1. × 10^13
5. × 10^12
0
0
2. × 10^-9
4. × 10^-9
6. × 10^-9
8. × 10^-9
1. × 10^-8
r


n = 6
1.6 × 10^13
1.4 × 10^13
1.2 × 10^13
1. × 10^13
8. × 10^12
6. × 10^12
4. × 10^12
2. × 10^12
0
0
2. × 10^-9
4. × 10^-9
6. × 10^-9
8. × 10^-9
1. × 10^-8
r


n = 12
2. × 10^12
1.5 × 10^12
1. × 10^12
5. × 10^11
0
0
2. × 10^-9
4. × 10^-9
6. × 10^-9
8. × 10^-9
1. × 10^-8
r