ANGULAR MOMENTUM COMMUTATORS IN HYDROGEN

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.48.

The generalized uncertainty principle we derived a long time ago can be applied to angular momentum operators as well. The principle is

$$\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\left\langle[\hat{A},\hat{B}]\right\rangle\right)^2 \tag{1}$$

For the particular case of $A = x^2$ and $B = L_z$, we have

$$\sigma_{x^2}^2\sigma_{L_z}^2 \geq \left(\frac{1}{2i}\langle[x^2,L_z]\rangle\right)^2 \tag{2}$$

Using $L_z = -i\hbar(x\partial/\partial y - y\partial/\partial x)$ and applying the commutator to a dummy function as usual, we find

$$[x^2,L_z] = -i\hbar\left[x^3\frac{\partial f}{\partial y} - x^2y\frac{\partial f}{\partial x} - x\frac{\partial\left(x^2f\right)}{\partial y} + y\frac{\partial\left(x^2f\right)}{\partial x}\right] \tag{3}$$

$$= -2xyi\hbar f \tag{4}$$

$$\sigma_{x^2}^2\sigma_{L_z}^2 \geq \hbar^2\langle xy\rangle^2 \tag{5}$$

Any ψ_{nlm} for the hydrogen atom is an eigenfunction of L_z with eigenvalue $\hbar m$ so the uncertainty in L_z for any such state is 0. Thus $\sigma_{L_z} = 0$. The conclusion from this is that $\langle xy\rangle = 0$ which can be verified by direct integration. The required integral is

$$\langle xy\rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}|\psi_{nlm}|^2(r\sin\theta\cos\phi)(r\sin\theta\sin\phi)r^2\sin\theta\, dr\, d\theta\, d\phi \tag{6}$$

Since $|\psi_{nlm}|^2$ is independent of ϕ (the spherical harmonics depend on ϕ only via a complex exponential which disappears when the square modulus is calculated) the ϕ integral is zero for any of the hydrogen wave functions.