

ANGULAR MOMENTUM COMMUTATORS IN HYDROGEN

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.48.

The generalized uncertainty principle we derived a long time ago can be applied to angular momentum operators as well. The principle is

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2 \quad (1)$$

For the particular case of $A = x^2$ and $B = L_z$, we have

$$\sigma_{x^2}^2 \sigma_{L_z}^2 \geq \left(\frac{1}{2i} \langle [x^2, L_z] \rangle \right)^2 \quad (2)$$

Using $L_z = -i\hbar(x\partial/\partial y - y\partial/\partial x)$ and applying the commutator to a dummy function as usual, we find

$$[x^2, L_z] = -i\hbar \left[x^3 \frac{\partial f}{\partial y} - x^2 y \frac{\partial f}{\partial x} - x \frac{\partial (x^2 f)}{\partial y} + y \frac{\partial (x^2 f)}{\partial x} \right] \quad (3)$$

$$= -2xyi\hbar f \quad (4)$$

$$\sigma_{x^2}^2 \sigma_{L_z}^2 \geq \hbar^2 \langle xy \rangle^2 \quad (5)$$

Any ψ_{nlm} for the hydrogen atom is an eigenfunction of L_z with eigenvalue $\hbar m$ so the uncertainty in L_z for any such state is 0. Thus $\sigma_{L_z} = 0$. The conclusion from this is that $\langle xy \rangle = 0$ which can be verified by direct integration. The required integral is

$$\langle xy \rangle = \int_0^{2\pi} \int_0^\pi \int_0^\infty |\psi_{nlm}|^2 (r \sin \theta \cos \phi)(r \sin \theta \sin \phi) r^2 \sin \theta dr d\theta d\phi \quad (6)$$

Since $|\psi_{nlm}|^2$ is independent of ϕ (the spherical harmonics depend on ϕ only via a complex exponential which disappears when the square modulus is calculated) the ϕ integral is zero for any of the hydrogen wave functions.