

SPIN 1/2: SPIN COMPONENTS

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.49.

As an example of some calculations with a spin 1/2 system, suppose we have an electron in a spin state given by

$$(1) \quad \chi = A \begin{pmatrix} 1 - 2i \\ 2 \end{pmatrix}$$

Requiring $|\chi|^2 = 1$ gives $|A|^2 (5 + 4) = 9|A|^2 = 1$, so if we take A as real, then $A = 1/3$.

Now suppose we measured each spin component on this system; what values could we get? Note that we're considering measuring a component, then restoring the system to its initial state, and then measuring the next component. If we didn't do a restoration after each measurement, we would alter the state to the eigenstate of the measured value, which we don't want to do.

First, we'll look at S_z . We need to express χ as a linear combination of the eigenspinors of S_z so:

$$(2) \quad \chi = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Comparison with the original state gives:

$$(3) \quad \chi = \frac{1 - 2i}{3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

so the possible values are $\hbar/2$ with probability $|(1 - 2i)/3|^2 = 5/9$ and $-\hbar/2$ with probability $4/9$.

To find the expectation value $\langle S_z \rangle$ we need to work out $\chi^T S_z \chi$.

$$(4) \quad \frac{1}{9} \begin{pmatrix} 1 + 2i & 2 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 - 2i \\ 2 \end{pmatrix} = \frac{\hbar}{18} (5 - 4) = \frac{\hbar}{18}$$

For S_x we use the eigenspinors from our earlier post

$$(5) \quad \chi = \frac{\sqrt{2}}{3} \left[a \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right]$$

Solving for a and b :

$$(6) \quad a + b = 1 - 2i$$

$$(7) \quad a - b = 2$$

$$(8) \quad a = \frac{3}{2} - i$$

$$(9) \quad b = -\frac{1}{2} - i$$

so

$$(10) \quad \chi = \frac{\sqrt{2}}{3} \left[\left(\frac{3}{2} - i \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left(-\frac{1}{2} - i \right) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right]$$

The possible values are $\hbar/2$ with probability $|(\sqrt{2}/3)(3/2 - i)|^2 = 13/18$ and $-\hbar/2$ with probability $5/18$.

The expectation value is found from the matrix $S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$(11) \quad \langle S_x \rangle = \frac{1}{9} \begin{pmatrix} 1 + 2i & 2 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 - 2i \\ 2 \end{pmatrix} = \frac{\hbar}{18} (2 + 4i + 2 - 4i) = \frac{2\hbar}{9}$$

For S_y , we can use the eigenspinors from before, and following the same procedure as above we get

$$(12) \quad \chi = \frac{\sqrt{2}}{3} \left[\left(\frac{1}{2} - 2i \right) \begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -i \end{pmatrix} \right]$$

The possible values are $\hbar/2$ with probability $|(\sqrt{2}/3)(1/2 - 2i)|^2 = 17/18$ and $-\hbar/2$ with probability $1/18$. The expectation value is

$$(13) \quad \langle S_y \rangle = \frac{1}{9} \begin{pmatrix} 1 + 2i & 2 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 - 2i \\ 2 \end{pmatrix} = \frac{\hbar}{18} (-2i + 4 + 2i + 4) = \frac{4\hbar}{9}$$

As a check, $\langle S_x \rangle^2 + \langle S_y \rangle^2 + \langle S_z \rangle^2 = (\hbar/2)^2$. Note that this isn't the same as $\langle S_x^2 \rangle + \langle S_y^2 \rangle + \langle S_z^2 \rangle = \langle S^2 \rangle = s(s+1)\hbar^2 = \frac{3\hbar^2}{4}$.

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