SPIN 1/2: SPIN COMPONENTS

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.49.

As an example of some calculations with a spin 1/2 system, suppose we have an electron in a spin state given by

$$\chi = A\begin{pmatrix} 1-2i \\ 2 \end{pmatrix} \tag{1}$$

Requiring $|\chi|^2 = 1$ gives $|A|^2(5+4) = 9|A|^2 = 1$, so if we take A as real, then $A = 1/3$.

Now suppose we measured each spin component on this system; what values could we get? Note that we're considering measuring a component, then restoring the system to its initial state, and then measuring the next component. If we didn't do a restoration after each measurement, we would alter the state to the eigenstate of the measured value, which we don't want to do.

First, we'll look at S_z. We need to express χ as a linear combination of the eigenspinors of S_z so:

$$\chi = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2}$$

Comparison with the original state gives:

$$\chi = \frac{1-2i}{3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3}$$

so the possible values are $\hbar/2$ with probability $|(1-2i)/3|^2 = 5/9$ and $-\hbar/2$ with probability 4/9.

To find the expectation value $\langle S_z\rangle$ we need to work out $\chi^T S_z \chi$.

$$\frac{1}{9}\begin{pmatrix} 1+2i & 2 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1-2i \\ 2 \end{pmatrix} = \frac{\hbar}{18}(5-4) = \frac{\hbar}{18} \tag{4}$$

For S_x we use the eigenspinors from our earlier post

$$\chi = \frac{\sqrt{2}}{3}\left[a\begin{pmatrix}1\\1\end{pmatrix}+b\begin{pmatrix}1\\-1\end{pmatrix}\right] \tag{5}$$

Solving for a and b:

$$\begin{aligned}
a+b &= 1-2i &(6)\\
a-b &= 2 &(7)\\
a &= \frac{3}{2}-i &(8)\\
b &= -\frac{1}{2}-i &(9)
\end{aligned}$$

so

$$\chi = \frac{\sqrt{2}}{3}\left[\left(\frac{3}{2}-i\right)\begin{pmatrix}1\\1\end{pmatrix}+\left(-\frac{1}{2}-i\right)\begin{pmatrix}1\\-1\end{pmatrix}\right] \tag{10}$$

The possible values are $\hbar/2$ with probability $|(\sqrt{2}/3)(3/2-i)|^2 = 13/18$ and $-\hbar/2$ with probability 5/18.

The expectation value is found from the matrix $S_x = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}$

$$\langle S_x\rangle = \frac{1}{9}\begin{pmatrix}1+2i & 2\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1-2i\\2\end{pmatrix} = \frac{\hbar}{18}(2+4i+2-4i) = \frac{2\hbar}{9} \tag{11}$$

For S_y we can use the eigenspinors from before, and following the same procedure as above we get

$$\chi = \frac{\sqrt{2}}{3}\left[\left(\frac{1}{2}-2i\right)\begin{pmatrix}1\\i\end{pmatrix}+\frac{1}{2}\begin{pmatrix}1\\-i\end{pmatrix}\right] \tag{12}$$

The possible values are $\hbar/2$ with probability $|(\sqrt{2}/3)(1/2-2i)|^2 = 17/18$ and $-\hbar/2$ with probability 1/18. The expectation value is

$$\langle S_y\rangle = \frac{1}{9}\begin{pmatrix}1+2i & 2\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1-2i\\2\end{pmatrix} = \frac{\hbar}{18}(-2i+4+2i+4) = \frac{4\hbar}{9} \tag{13}$$

As a check, $\langle S_x\rangle^2+\langle S_y\rangle^2+\langle S_z\rangle^2 = (\hbar/2)^2$. Note that this isn't the same as $\langle S_x^2\rangle+\langle S_y^2\rangle+\langle S_z^2\rangle = \langle S^2\rangle = s(s+1)\hbar^2 = \frac{3\hbar^2}{4}$.

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