

CLEBSCH-GORDAN COEFFICIENTS FOR HIGHER SPIN

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.51.

The Clebsch-Gordan coefficients tell us how to combine two spin states to get a state that is an eigenstate of the total spin operator squared S^2 . Calculating these coefficients for states with higher spin than $1/2$ can be tedious, but here's an example for the case where $s_1 = 1/2$ and s_2 is any other spin. That is, the total spin will be one of $s_2 \pm \frac{1}{2}$.

First, collect the equations we need from an earlier post:

$$(1) \quad S^2 |sm\rangle = s(s+1) |sm\rangle$$

$$(2) \quad S_{\pm} |sm\rangle = \hbar \sqrt{s(s+1) - m(m \pm 1)} |sm \pm 1\rangle$$

$$(3) \quad S_x = \frac{1}{2} (S_+ + S_-)$$

$$(4) \quad S_y = \frac{1}{2i} (S_+ - S_-)$$

$$(5) \quad S_z |sm\rangle = m\hbar |sm\rangle$$

The general problem is to find A and B such that the following state is an eigenstate of S^2 :

$$(6) \quad |sm\rangle = A \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| s_2 m - \frac{1}{2} \right\rangle + B \left| \frac{1}{2} - \frac{1}{2} \right\rangle \left| s_2 m + \frac{1}{2} \right\rangle$$

That is, the total z component is m , and can be formed by a combination of the spin $1/2$ state with $s_z = +\frac{1}{2}$ and the spin s_2 state with $s_z = m - \frac{1}{2}$, or by a combination of the spin $1/2$ state with $s_z = -\frac{1}{2}$ and the spin s_2 state with $s_z = m + \frac{1}{2}$.

We expand the S^2 operator as before:

$$(7) \quad S^2 = (S^{(1)})^2 + (S^{(2)})^2 + 2\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}$$

where

$$(8) \quad \mathbf{S}^{(1,2)} = [S_x^{(1,2)}, S_y^{(1,2)}, S_z^{(1,2)}]$$

is the spin operator for each of the two component spins.

The last term in 7 thus expands to

$$(9) \quad \mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)} = S_x^{(1)} S_x^{(2)} + S_y^{(1)} S_y^{(2)} + S_z^{(1)} S_z^{(2)}$$

The actions of the various S^2 and S_z operators is easy enough, as the states in 6 are eigenstates of these operators. The operators involving S_x and S_y can be worked out using equations 2 to 4. We get

$$(10) \quad S_x^{(1)} S_x^{(2)} \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| s_2 m - \frac{1}{2} \right\rangle = \sqrt{s_2(s_2+1) - (m+1/2)(m-1/2)} \left| \frac{1}{2} - \frac{1}{2} \right\rangle \left| s_2 m + \frac{1}{2} \right\rangle$$

with similar expressions for the other cases.

Applying 7 to 6 and using the relations above as required, we get, after collecting terms

$$(11) \quad \frac{1}{\hbar^2} S^2 |sm\rangle = S^2 \left(A \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| s_2 m - \frac{1}{2} \right\rangle + B \left| \frac{1}{2} - \frac{1}{2} \right\rangle \left| s_2 m + \frac{1}{2} \right\rangle \right)$$

$$(12) \quad = ((S^{(1)})^2 + (S^{(2)})^2 + 2\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}) \left(A \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| s_2 m - \frac{1}{2} \right\rangle + B \left| \frac{1}{2} - \frac{1}{2} \right\rangle \left| s_2 m + \frac{1}{2} \right\rangle \right)$$

$$(13) \quad = A \left(\frac{3}{4} + s_2(s_2+1) + \left(m - \frac{1}{2} \right) \right) \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| s_2 m - \frac{1}{2} \right\rangle +$$

$$(14) \quad B \sqrt{s_2(s_2+1) - (m+1/2)(m-1/2)} \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| s_2 m - \frac{1}{2} \right\rangle +$$

$$(15) \quad A \sqrt{s_2(s_2+1) - (m+1/2)(m-1/2)} \left| \frac{1}{2} - \frac{1}{2} \right\rangle \left| s_2 m + \frac{1}{2} \right\rangle +$$

$$(16) \quad B \left(\frac{3}{4} + s_2(s_2+1) - \left(m + \frac{1}{2} \right) \right) \left| \frac{1}{2} - \frac{1}{2} \right\rangle \left| s_2 m + \frac{1}{2} \right\rangle$$

Other terms involving the states $|s_2 m + \frac{3}{2}\rangle$ and $|s_2 m - \frac{3}{2}\rangle$ are zero, as the raising and lowering operators cannot produce states outside the range $m \pm \frac{1}{2}$ (the term in 2 is zero in these cases).

From 1 and 6 and the fact that the total spin s must be $s_2 \pm \frac{1}{2}$ we must have

(17)

$$\frac{1}{\hbar^2} S^2 |sm\rangle = \left(s_2 \pm \frac{1}{2}\right) \left(s_2 + 1 \pm \frac{1}{2}\right) |sm\rangle$$

(18)

$$= \left(s_2 \pm \frac{1}{2}\right) \left(s_2 + 1 \pm \frac{1}{2}\right) \left(A \left| \frac{1}{2} \frac{1}{2} \right\rangle |s_2 m - \frac{1}{2}\rangle + B \left| \frac{1}{2} - \frac{1}{2} \right\rangle |s_2 m + \frac{1}{2}\rangle \right)$$

The coefficients of each of the two states must be equal on each side of the equation, so we get

(19)

$$A \left(\frac{3}{4} + s_2(s_2 + 1) + \left(m - \frac{1}{2}\right) \right) + B \sqrt{s_2(s_2 + 1) - \left(m + \frac{1}{2}\right)\left(m - \frac{1}{2}\right)} = A \left(s_2 \pm \frac{1}{2}\right) \left(s_2 + 1 \pm \frac{1}{2}\right)$$

(20)

$$B \left(\frac{3}{4} + s_2(s_2 + 1) - \left(m + \frac{1}{2}\right) \right) + A \sqrt{s_2(s_2 + 1) - \left(m + \frac{1}{2}\right)\left(m - \frac{1}{2}\right)} = B \left(s_2 \pm \frac{1}{2}\right) \left(s_2 + 1 \pm \frac{1}{2}\right)$$

These equations have the form

$$(21) \quad a_1 A + b B = c A$$

$$(22) \quad a_2 B + b A = c B$$

We can try to solve them by noting from the second equation

$$(23) \quad A = B \frac{(c - a_2)}{b}$$

Substituting this back into the first equation just gives an identity, since the factor of B cancels out, so we can get only a relation between A and B .

$$(24) \quad A = B \frac{\left(s_2 \pm \frac{1}{2}\right)\left(s_2 + 1 \pm \frac{1}{2}\right) - s_2(s_2 + 1) - 1/4 + m}{\sqrt{s_2(s_2 + 1) - \left(m + \frac{1}{2}\right)\left(m - \frac{1}{2}\right)}}$$

To determine A and B independently, we must impose the normalization condition on the states, so we must have

$$(25) \quad A^2 + B^2 = 1$$

$$(26) \quad \left\{ \left[\frac{(s_2 \pm \frac{1}{2})(s_2 + 1 \pm \frac{1}{2}) - s_2(s_2 + 1) - 1/4 + m}{\sqrt{s_2(s_2 + 1) - (m + \frac{1}{2})(m - \frac{1}{2})}} \right]^2 + 1 \right\} B^2 = 1$$

$$(27) \quad \frac{((s_2 \pm \frac{1}{2})(s_2 + 1 \pm \frac{1}{2}) - s_2(s_2 + 1) - 1/4 + m)^2 + s_2(s_2 + 1) - (m + \frac{1}{2})(m - \frac{1}{2})}{s_2(s_2 + 1) - (m + \frac{1}{2})(m - \frac{1}{2})} B^2 = 1$$

The numerator comes out to

$$(28) \quad 2s_2^2 + s_2 \left(\pm 2m \mp \frac{1}{2} + 1 \pm \frac{1}{2} + 1 \right) + m \left(-\frac{1}{2} \pm 1 + \frac{1}{2} \right) + \left(\mp \frac{1}{4} + \frac{5}{16} + \frac{1}{8} \pm \frac{1}{4} + \frac{1}{16} \right) =$$

$$(29) \quad 2s_2^2 + 2s_2(1 \pm m) \pm m + \frac{1}{2} =$$

$$(30) \quad 2s_2(s_2 + 1) + \frac{1}{2} \pm m(2s_2 + 1)$$

Thus we get

$$(31) \quad B^2 = \frac{s_2(s_2 + 1) - (m + \frac{1}{2})(m - \frac{1}{2})}{2s_2(s_2 + 1) + \frac{1}{2} \pm m(2s_2 + 1)}$$

$$(32) \quad = \frac{(s_2 + \frac{1}{2} \mp m)(s_2 + \frac{1}{2} \pm m)}{(2s_2 + 1)(s_2 + \frac{1}{2} \pm m)}$$

$$(33) \quad = \frac{s_2 + \frac{1}{2} \mp m}{2s_2 + 1}$$

$$(34) \quad B = \pm \sqrt{\frac{s_2 + \frac{1}{2} \mp m}{2s_2 + 1}}$$

$$(35) \quad A = \sqrt{1 - B^2}$$

$$(36) \quad = \sqrt{\frac{s_2 + \frac{1}{2} \pm m}{2s_2 + 1}}$$

In the second line, the denominator is equal to the denominator in the first line as can be seen by direct multiplication of both. The numerator in the second line is

$$(37) \quad (s_2 + \frac{1}{2} \mp m)(s_2 + \frac{1}{2} \pm m) = \left(s_2 + \frac{1}{2}\right)^2 - m^2$$

$$(38) \quad = s_2^2 + s_2 + \frac{1}{4} - m^2$$

which is equal to the numerator in the first line.

We can check that these equations work in the original case of $s_2 = \frac{1}{2}$. If $s = s_2 + \frac{1}{2} = 1$ and $m = 0$, then $B = 1/\sqrt{2} = A$. If $m = 1$, then $A = 1$ and $B = 0$, while if $m = -1$ then $A = 0$ and $B = 1$. If $s = s_2 - \frac{1}{2} = 0$, then $A = 1/\sqrt{2}$, $B = -1/\sqrt{2}$.

For a more complex example, suppose $s_2 = 2$. Then if $s = s_2 + \frac{1}{2} = 5/2$ and $m = 3/2$, we get $A = \sqrt{4/5}$ and $B = \sqrt{1/5}$. This matches the entry in standard tables.

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