

SPIN MATRICES: GENERAL CASE

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.53.

We can generalize the calculation for spin 3/2 to get the spin matrices for any spin s .

We first find S_{\pm} from the equation

$$(0.1) \quad S_{\pm}|sm\rangle = \hbar\sqrt{s(s+1) - m(m\pm 1)}|sm\pm 1\rangle$$

First, consider S_+ . We know that $S_+|ss\rangle = 0$ and that $m = s - k$ for $k = 1 \dots 2s$ for the remaining eigenstates of S_z . We can also see from the spin 3/2 case that the S_+ matrix contains non-zero entries only on the diagonal above the main diagonal (since the raising operator always raises an eigenstate m of S_z to the state $m + 1$), and that these entries consist of the square root factor given in 0.1 for values of m starting with $m = s - 1$ as matrix element $S_{+(1,2)}$ (i.e. the element in row 1, column 2 of matrix S_+), and then working down the diagonal, with element $S_{+(k,k+1)}$ being the square root factor for $m = s - k$.

Letting $m = s - k$ for $k = 1 \dots 2s$ we get

$$(0.2) \quad \sqrt{s(s+1) - m(m+1)} = \sqrt{s(s+1) - (s-k)(s-k+1)}$$

$$(0.3) \quad = \sqrt{2sk + k(1-k)}$$

$$(0.4) \quad \equiv c_k$$

We can therefore write

$$(0.5) \quad S_+ = \hbar \begin{bmatrix} 0 & c_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & c_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & c_3 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & c_{2s} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

For those readers wanting to get the answer as given in Griffiths's book, we can make the substitution

$$(0.6) \quad j \equiv -k + 1 + s$$

so that $j = s \dots - s + 1$ as $k = 1 \dots 2s$. Then

$$(0.7) \quad \sqrt{2sk + k(1-k)} = \sqrt{2(1+s-j)s + (1+s-j)(j-s)}$$

$$(0.8) \quad = \sqrt{(1+s-j)(j+s)}$$

$$(0.9) \quad \equiv b_j$$

as given in the book.

The calculation for S_- is similar, giving $S_{-(k+1,k)} = c_k$ for $k = 1 \dots 2s$, so we get

$$(0.10) \quad S_- = \hbar \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ c_1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & c_2 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & c_3 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & c_{2s} & 0 \end{bmatrix}$$

In Griffiths's notation, $S_{-(k+1,k)} = b_j$, where $j = -k + 1 + s$ as before, and we can then use the formulas $S_x = (S_+ + S_-)/2$ and $S_y = (S_+ - S_-)/2i$ to get the actual spin matrices, which will have non-zero entries on the two diagonals on either side of the main diagonal.