SPIN MATRICES: GENERAL CASE

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.53.

We can generalize the calculation for spin 3/2 to get the spin matrices for any spin s.

We first find $S_{\pm}$ from the equation

$$S_{\pm}|s\,m\rangle=\hbar\sqrt{s(s+1)-m(m\pm1)}|s\,m\pm1\rangle \tag{1}$$

First, consider S_{+}. We know that $S_{+}|s\,s\rangle=0$ and that $m=s-k$ for $k=1\ldots2s$ for the remaining eigenstates of S_z. We can also see from the spin 3/2 case that the S_{+}matrix contains non-zero entries only on the diagonal above the main diagonal (since the raising operator always raises an eigenstate m of S_z to the state $m+1$), and that these entries consist of the square root factor given in 1 for values of m starting with $m=s-1$ as matrix element $S_{+(1,2)}$ (i.e. the element in row 1, column 2 of matrix S_{+}), and then working down the diagonal, with element $S_{+(k,k+1)}$ being the square root factor for $m=s-k$.

Letting $m=s-k$ for $k=1\ldots2s$ we get

$$\begin{aligned}
\sqrt{s(s+1)-m(m+1)} &= \sqrt{s(s+1)-(s-k)(s-k+1)} && (2)\\
&= \sqrt{2sk+k(1-k)} && (3)\\
&\equiv c_k && (4)
\end{aligned}$$

We can therefore write

$$S_{+}=\hbar\left[\begin{array}{ccccccc}
0 & c_1 & 0 & 0 & \ldots & 0 & 0\\
0 & 0 & c_2 & 0 & \ldots & 0 & 0\\
0 & 0 & 0 & c_3 & \ldots & 0 & 0\\
0 & 0 & 0 & 0 & \ldots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \ldots & 0 & c_{2s}\\
0 & 0 & 0 & 0 & \ldots & 0 & 0
\end{array}\right] \tag{5}$$

For those readers wanting to get the answer as given in Griffiths's book, we can make the substitution

$$j\equiv -k+1+s \tag{6}$$

so that $j=s\ldots-s+1$ as $k=1\ldots2s$. Then

$$\begin{aligned}
\sqrt{2sk+k(1-k)} &= \sqrt{2(1+s-j)s+(1+s-j)(j-s)} && (7)\\
&= \sqrt{(1+s-j)(j+s)} && (8)\\
&\equiv b_j && (9)
\end{aligned}$$

as given in the book.

The calculation for S_- is similar, giving $S_{-(k+1,k)}=c_k$ for $k=1\ldots2s$, so we get

$$S_-=\hbar\begin{bmatrix} 0 & 0 & 0 & 0 & \ldots & 0 & 0\\ c_1 & 0 & 0 & 0 & \ldots & 0 & 0\\ 0 & c_2 & 0 & 0 & \ldots & 0 & 0\\ 0 & 0 & c_3 & 0 & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \ldots & 0 & 0\\ 0 & 0 & 0 & 0 & \ldots & c_{2s} & 0 \end{bmatrix} \tag{10}$$

In Griffiths's notation, $S_{-(k+1,k)}=b_j$, where $j=-k+1+s$ as before, and we can then use the formulas $S_x=(S_++S_-)/2$ and $S_y=(S_+-S_-)/2i$ to get the actual spin matrices, which will have non-zero entries on the two diagonals on either side of the main diagonal.