SPIN MATRICES: GENERAL CASE

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.53.

We can generalize the calculation for spin 3/2 to get the spin matrices for any spin s.

We first find $S_{\pm}$ from the equation

$$S_{\pm}|s\,m\rangle = \hbar\sqrt{s(s+1)-m(m\pm 1)}|s\,m\pm 1\rangle \tag{1}$$

First, consider S_+. We know that $S_+|s\,s\rangle = 0$ and that $m = s-k$ for $k = 1\ldots 2s$ for the remaining eigenstates of S_z. We can also see from the spin 3/2 case that the S_+matrix contains non-zero entries only on the diagonal above the main diagonal (since the raising operator always raises an eigenstate m of S_z to the state $m+1$), and that these entries consist of the square root factor given in 1 for values of m starting with $m = s-1$ as matrix element $S_{+(1,2)}$ (i.e. the element in row 1, column 2 of matrix S_+), and then working down the diagonal, with element $S_{+(k,k+1)}$ being the square root factor for $m = s-k$.

Letting $m = s-k$ for $k = 1\ldots 2s$ we get

$$\begin{aligned}
\sqrt{s(s+1)-m(m+1)} &= \sqrt{s(s+1)-(s-k)(s-k+1)} && (2)\\
&= \sqrt{2sk+k(1-k)} && (3)\\
&\equiv c_k && (4)
\end{aligned}$$

We can therefore write

$$S_+ = \hbar\begin{bmatrix}
0 & c_1 & 0 & 0 & \ldots & 0 & 0\\
0 & 0 & c_2 & 0 & \ldots & 0 & 0\\
0 & 0 & 0 & c_3 & \ldots & 0 & 0\\
0 & 0 & 0 & 0 & \ldots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \ldots & 0 & c_{2s}\\
0 & 0 & 0 & 0 & \ldots & 0 & 0
\end{bmatrix} \tag{5}$$

For those readers wanting to get the answer as given in Griffiths's book, we can make the substitution

$$j \equiv -k+1+s \tag{6}$$

so that $j = s \ldots -s+1$ as $k = 1 \ldots 2s$. Then

$$\begin{aligned}
\sqrt{2sk+k(1-k)} &= \sqrt{2(1+s-j)s+(1+s-j)(j-s)} && (7)\\
&= \sqrt{(1+s-j)(j+s)} && (8)\\
&\equiv b_j && (9)
\end{aligned}$$

as given in the book.

The calculation for S_- is similar, giving $S_{-(k+1,k)} = c_k$ for $k = 1 \ldots 2s$, so we get

$$S_- = \hbar \begin{bmatrix} 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ c_1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & c_2 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & c_3 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & c_{2s} & 0 \end{bmatrix} \tag{10}$$

In Griffiths's notation, $S_{-(k+1,k)} = b_j$, where $j = -k+1+s$ as before, and we can then use the formulas $S_x = (S_+ + S_-)/2$ and $S_y = (S_+ - S_-)/2i$ to get the actual spin matrices, which will have non-zero entries on the two diagonals on either side of the main diagonal.