

## ANGULAR MOMENTUM AS A GENERATOR OF ROTATIONS

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.56.

An interesting property of the operator  $L_z$  is that it can act as a *generator of rotations* about the  $z$  axis.

Using the series expansion of the exponential, and the form of  $L_z$  in spherical coordinates,  $L_z = (\hbar/i)\partial/\partial\phi$ , we get

$$(1) \quad e^{iL_z\varphi/\hbar}f(\phi) = \sum_{j=0}^{\infty} \frac{\varphi^j}{j!} \frac{\partial^j f}{\partial\phi^j}$$

which is the Taylor series for  $f(\phi + \varphi)$ . Thus the operator  $e^{iL_z\varphi/\hbar}$  effectively rotates  $f(\phi)$  through an angle  $\varphi$ .

In general,  $e^{i\mathbf{L}\cdot\hat{n}\varphi/\hbar}$  is an operator that will rotate a function through an angle  $\varphi$  about the axis  $\hat{n}$ .

The use of  $\mathbf{L}$  causes rotations in ordinary 3-d space. If we want to rotate spinors, we can use the spin operator  $\mathbf{S}$ , and in the case of spin 1/2, we can use the Pauli matrices to produce the operator  $e^{i\boldsymbol{\sigma}\cdot\hat{n}\varphi/2}$ , which will rotate a spin 1/2 spinor  $\chi_{\pm}$ .

To see what this means, we can work out the exponential in a more convenient form. We start with

$$(2) \quad \hat{n} \cdot \boldsymbol{\sigma} = \sigma_x \sin \theta \cos \phi + \sigma_y \sin \theta \sin \phi + \sigma_z \cos \theta$$

Substituting the spin matrices, we get

$$(3) \quad \hat{n} \cdot \boldsymbol{\sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sin \theta \cos \phi + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sin \theta \sin \phi + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cos \theta = \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix}$$

Note that (by direct multiplication):

$$(4) \quad (\hat{n} \cdot \boldsymbol{\sigma})^{2j} = I$$

$$(5) \quad (\hat{n} \cdot \boldsymbol{\sigma})^{2j+1} = \hat{n} \cdot \boldsymbol{\sigma}$$

where  $j = 0, 1, 2, 3, \dots$ . That is, all even powers of  $\hat{n} \cdot \sigma$  are the identity matrix  $I$  and all odd powers are  $\hat{n} \cdot \sigma$  itself. We can plug this into the expression  $e^{i\sigma \cdot \hat{n}\varphi/2}$  for spinor rotations and use the series expansion of the exponential:

(6)

$$e^{i(\hat{n} \cdot \sigma)\varphi/2} = \sum_{j=0}^{\infty} \frac{(i(\hat{n} \cdot \sigma)\varphi/2)^j}{j!}$$

(7)

$$= \sum_{j=0}^{\infty} \frac{(i\varphi/2)^{2j}}{(2j)!} + (\hat{n} \cdot \sigma) \sum_{j=0}^{\infty} \frac{(i\varphi/2)^{2j+1}}{(2j+1)!}$$

(8)

$$= 1 - \frac{(\varphi/2)^2}{2!} + \frac{(\varphi/2)^4}{4!} - \dots + i(\hat{n} \cdot \sigma) \left[ \frac{(\varphi/2)}{1!} - \frac{(\varphi/2)^3}{3!} + \frac{(\varphi/2)^5}{5!} - \dots \right]$$

(9)

$$= \cos(\varphi/2) + i(\hat{n} \cdot \sigma) \sin(\varphi/2)$$

where we have used the standard series expansions for cos and sin to get the last line.

If the axis of rotation is the  $x$ -axis, then  $\hat{n} = [1, 0, 0]$  and  $\hat{n} \cdot \sigma = \sigma_x$  so for a rotation of  $\varphi = \pi$  we get for the rotation matrix  $R$ :

$$(10) \quad R = e^{i\sigma \cdot \hat{n}\varphi/2} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\sigma_x$$

which swaps  $\chi_+$  and  $\chi_-$ ; that is, it converts spin up into spin down, and vice versa, as you would expect. The extra factor of  $i$  is a phase shift in the wave function and can produce interference effects between particles.

With  $\hat{n} = [0, 1, 0]$  and  $\varphi = \pi/2$  we get  $\hat{n} \cdot \sigma = \sigma_y$

$$(11) \quad R = \frac{\sqrt{2}}{2}(I + i\sigma_y) = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

When applied to  $\chi_+$  we get

$$(12) \quad R\chi_+ = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

This is an eigenspinor of  $\sigma_x$  which again is what you'd expect, since the rotation rotates the  $z$  axis into the  $x$  axis.

With  $\hat{n} = [0, 0, 1]$  and  $\varphi = 2\pi$  we get  $\hat{n} \cdot \sigma = \sigma_z$

$$(13) \quad R = -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

The fact that a rotation through  $2\pi$  produces a factor of  $-1$  is another phase shift effect, as in the first example above, and does actually produce interference effects, for example, when experiments involving rotation in a magnetic field are done.

### Extra bit

Irrelevant to the question, but a cool proof so I thought I'd include it anyway.

The  $k$ th derivative of  $x^n f(x)$  is given by

$$(14) \quad \frac{d^k}{dx^k}(x^n f(x)) = \sum_{j=0}^k \frac{n!}{(n-j)!} \binom{k}{j} x^{n-j} f^{(k-j)}$$

where  $\binom{k}{j} = \frac{k!}{j!(k-j)!}$  is the binomial coefficient, and  $f^{(k-j)}$  is the  $(k-j)$ th derivative of  $f$ .

We can prove this by induction. First, we prove the anchor step, for  $k=0$ . From this equation with  $k=0$  both sides of the equation give  $x^n f(x)$  so the formula is valid here.

Next, we assume the above equation is valid for  $k$  and prove this implies it is valid also for  $k+1$ . Taking the derivative of both sides gives

$$(15) \quad \frac{d^{k+1}}{dx^{k+1}}(x^n f(x)) = \sum_{j=0}^k \frac{n!}{(n-j)!} \binom{k}{j} (n-j)x^{n-j-1} f^{(k-j)} + \sum_{j=0}^k \frac{n!}{(n-j)!} \binom{k}{j} (n-j)x^{n-j} f^{(k-j+1)}$$

$$(16) \quad = \frac{n!}{(n-k)!} (n-k)x^{n-k-1} f +$$

$$(17) \quad \sum_{j=1}^k x^{n-j} f^{(k-j+1)} \left[ \frac{n!}{(n-j+1)!} \binom{k}{j-1} (n-j+1) + \frac{n!}{(n-j)!} \binom{k}{j} \right] + x^n f^{(k+1)}$$

$$(18) \quad = \frac{n!}{(n-k-1)!} x^{n-k-1} f + \sum_{j=1}^k x^{n-j} f^{(k-j+1)} \frac{n!}{(n-j)!} \left[ \binom{k}{j-1} + \binom{k}{j} \right] + x^n f^{(k+1)}$$

$$(19) \quad = \sum_{j=0}^{k+1} \frac{n!}{(n-j)!} \binom{k+1}{j} x^{n-j} f^{(k+1-j)}$$

In going from step 1 to step 2, we have separated out the  $j = k$  term from the first sum and the  $j = 0$  term from the second sum. Then we replaced  $j$  by  $j - 1$  in the first sum so we could group together common powers of  $x$  in the two sums.

The last step uses the formula

$$(20) \quad \binom{k}{j-1} + \binom{k}{j} = \binom{k+1}{j}$$

which can be proved by putting the LHS over a common denominator and adding.