ANGULAR MOMENTUM: RESTRICTION TO INTEGER VALUES

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.57.

When working out the possible eigenvalues for the angular momentum operators, we found that angular momentum (and its 3 components) can take on integer or half-integer multiples of $\hbar$. When we found the eigenfunctions of the orbital angular momentum operators, however, we discovered that they are spherical harmonics, and therefore only integer multiples of $\hbar$ are allowed as eigenvalues.

A curious way of analyzing this problem goes as follows. Suppose we define a set of operators as follows:

$$q_1 \equiv \frac{1}{\sqrt{2}}\left[x+\frac{a^2}{\hbar}p_y\right] \tag{1}$$

$$q_2 \equiv \frac{1}{\sqrt{2}}\left[x-\frac{a^2}{\hbar}p_y\right] \tag{2}$$

$$p_1 \equiv \frac{1}{\sqrt{2}}\left[p_x-\frac{\hbar}{a^2}y\right] \tag{3}$$

$$p_2 \equiv \frac{1}{\sqrt{2}}\left[p_x+\frac{\hbar}{a^2}y\right] \tag{4}$$

where p_x and p_y are components of linear momentum and a is the Bohr length in hydrogen, as usual.

Since $[x,p_y]=[y,p_x]=0$, it follows that $[q_1,q_2]=[p_1,p_2]=0$. Also, since $[x,p_x]=[y,p_y]=i\hbar$, it follows that $[q_1,p_1]=[q_2,p_2]=i\hbar$. These new operators thus satisfy the same commutation relations as the traditional position and linear momentum operators.

Now we can express L_z in terms of these new operators by observing

$$q_{1,2}^2 = \frac{1}{2}\left[x^2\pm 2\frac{a^2}{\hbar}xp_y+\frac{a^4}{\hbar^2}p_y^2\right] \tag{5}$$

$$p_{1,2}^2 = \frac{1}{2}\left[p_x^2\mp 2\frac{\hbar}{a^2}yp_x+\frac{\hbar^2}{a^4}y^2\right] \tag{6}$$

Combining these, we get

$$\frac{\hbar}{2a^2}(q_1^2-q_2^2)+\frac{a^2}{2\hbar}(p_1^2-p_2^2) = xp_y - yp_x \tag{7}$$

$$= L_z \tag{8}$$

The Hamiltonian for a harmonic oscillator is $H = p^2/2m + m\omega^2 r^2/2$. If $m = \hbar/a^2$ and $\omega = 1$ then

$$H = \frac{p^2a^2}{2\hbar} + \frac{\hbar r^2}{2a^2} \tag{9}$$

If we take the momentums of the two oscillators to be $p_{1,2}$ and the positions to be $q_{1,2}$ then we have

$$H_{1,2} = \frac{p_{1,2}^2a^2}{2\hbar} + \frac{\hbar q_{1,2}^2}{2a^2} \tag{10}$$

Using the relation above, we get

$$H_1 - H_2 = L_z \tag{11}$$

It should be pointed out here that the units don't match up in this equation, since we have energy on the left and angular momentum on the right. This is because we've taken $\hbar/a^2$ as a mass, when in fact its units are mass/time. We've also taken ω as dimensionless, whereas a true frequency has dimensions of 1/time. The point here, though, is not to treat $H_{1,2}$ as actual energies; rather we consider the purely mathematical problem of finding the eigenvalues of $H_{1,2}$.

Since the operators $H_{1,2}$ have the form of the harmonic oscillator hamiltonian, we know that their eigenvalues are $\left(n_{1,2}+\frac{1}{2}\right)\hbar\omega$. Since $L_z = H_1 - H_2$, the eigenvalues of L_z must be of form $m\hbar = (n_1+1/2)\hbar\omega - (n_2+1/2)\hbar\omega$, so since $\omega = 1$, $m = n_1 - n_2$ which must be an integer.

It's not entirely clear to me how this analysis gives any more insight into the question of why the half-integer eigenvalues aren't allowed for orbital angular momentum, but at least the results are consistent with the spherical harmonic solution.