SPIN 1/2: MINIMUM UNCERTAINTY

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.58.

For spin 1/2, we can work out the uncertainty relation for the spin components. if we have a general spin 1/2 state, it can be written as

$$\chi = a\chi_+ + b\chi_- \tag{1}$$

where $\chi_{+,-}$ are the eigenspinors of S_z and a and b are complex constants.

Using the results on spin expectation values, we have

$$\begin{aligned}
\sigma_x^2 &= \langle S_x^2\rangle - \langle S_x\rangle^2 &(2)\\
&= \frac{\hbar^2}{4}(1-(a^*b-ab^*)^2) &(3)\\
\sigma_y^2 &= \langle S_y^2\rangle - \langle S_y\rangle^2 &(4)\\
&= \frac{\hbar^2}{4}(1-(-a^*bi+ab^*i)^2) &(5)
\end{aligned}$$

The normalization condition

$$|a|^2+|b|^2 = 1 \tag{6}$$

means that we if $a = a_r e^{i\theta_a}$and $b = b_r e^{i\theta_b}$ we can transfer the phase of a to b by redefining $a = a_r$ and $b = b_r e^{i(\theta_b-\theta_a)}$ without violating either the relative phase between a and b or the normalization condition.

So, taking a to be real, and $b = b_r + ib_i$

$$\begin{aligned}
\sigma_x^2\sigma_y^2 &= \frac{\hbar^4}{16}(1-(ab-ab^*)^2)(1-(-abi+ab^*i)^2) &(7)\\
&= \frac{\hbar^4}{16}(1-4a^2b_r^2)(1-4a^2b_i^2) &(8)\\
&= \frac{\hbar^4}{16}(1-4a^2(b_r^2+b_i^2)+16a^4b_r^2(1-a^2-b_r^2)) &(9)\\
&= \frac{\hbar^4}{16}(1-4a^2(1-a^2)+16a^4b_r^2(1-a^2-b_r^2)) &(10)\\
&= \frac{\hbar^4}{16}((1-2a^2)^2+16a^4b_r^2(1-a^2-b_r^2)) &(11)
\end{aligned}$$

In the fourth line, we used the normalization condition 6.

The term $16a^4b_r^2(1-a^2-b_r^2)$ has the form of a parabola such as $Ax(B-x)$ with $A=16a^4$, $B=1-a^2$ and $x=b_r^2$. Since b_r^2 is restricted to the interval $[0,1-a^2]$ we see that this portion of a parabola has its minimum values when $b_r^2=0$, $1-a^2$ and its maximum value at the midpoint of the interval, with $b_r^2=(1-a^2)/2$. Thus the minimum value of $\sigma_x^2\sigma_y^2$ occurs when $b_r^2=0$, $1-a^2$. If $b_r^2=0$, b is imaginary, while if $b_r^2=1-a^2$, $b_i=0$ and b is real. In either case we have

$$\sigma_x\sigma_y=\frac{\hbar^2}{4}|(1-2a^2)|=\frac{\hbar^2}{4}|(|b|^2-a^2)|=\frac{\hbar}{2}|\langle S_z\rangle| \tag{12}$$

From the general uncertainty principle, we have

$$\begin{aligned}
\sigma_x^2\sigma_y^2 &\geq \left(\frac{1}{2i}\left\langle[S_x,S_y]\right\rangle\right)^2 &(13)\\
&= \frac{\hbar^2}{4}|\langle S_z\rangle|^2 &(14)
\end{aligned}$$

Thus the minimum uncertainty for the spin 1/2 state is actually the lower limit. We have used absolute values when taking the square root, since uncertainties must be positive.