

SPIN 1/2: MINIMUM UNCERTAINTY

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.58.

For spin 1/2, we can work out the uncertainty relation for the spin components. if we have a general spin 1/2 state, it can be written as

$$(1) \quad \chi = a\chi_+ + b\chi_-$$

where $\chi_{+,-}$ are the eigenspinors of S_z and a and b are complex constants.

Using the results on spin expectation values, we have

$$(2) \quad \sigma_x^2 = \langle S_x^2 \rangle - \langle S_x \rangle^2$$
$$(3) \quad = \frac{\hbar^2}{4} (1 - (a^*b - ab^*)^2)$$
$$(4) \quad \sigma_y^2 = \langle S_y^2 \rangle - \langle S_y \rangle^2$$
$$(5) \quad = \frac{\hbar^2}{4} (1 - (-a^*bi + ab^*i)^2)$$

The normalization condition

$$(6) \quad |a|^2 + |b|^2 = 1$$

means that we if $a = a_r e^{i\theta_a}$ and $b = b_r e^{i\theta_b}$ we can transfer the phase of a to b by redefining $a = a_r$ and $b = b_r e^{i(\theta_b - \theta_a)}$ without violating either the relative phase between a and b or the normalization condition.

So, taking a to be real, and $b = b_r + ib_i$

$$\begin{aligned}
(7) \quad \sigma_x^2 \sigma_y^2 &= \frac{\hbar^4}{16} (1 - (ab - ab^*)^2) (1 - (-abi + ab^*i)^2) \\
(8) \quad &= \frac{\hbar^4}{16} (1 - 4a^2 b_r^2) (1 - 4a^2 b_i^2) \\
(9) \quad &= \frac{\hbar^4}{16} (1 - 4a^2 (b_r^2 + b_i^2) + 16a^4 b_r^2 (1 - a^2 - b_r^2)) \\
(10) \quad &= \frac{\hbar^4}{16} (1 - 4a^2 (1 - a^2) + 16a^4 b_r^2 (1 - a^2 - b_r^2)) \\
(11) \quad &= \frac{\hbar^4}{16} ((1 - 2a^2)^2 + 16a^4 b_r^2 (1 - a^2 - b_r^2))
\end{aligned}$$

In the fourth line, we used the normalization condition 6.

The term $16a^4 b_r^2 (1 - a^2 - b_r^2)$ has the form of a parabola such as $Ax(B - x)$ with $A = 16a^4$, $B = 1 - a^2$ and $x = b_r^2$. Since b_r^2 is restricted to the interval $[0, 1 - a^2]$ we see that this portion of a parabola has its minimum values when $b_r^2 = 0, 1 - a^2$ and its maximum value at the midpoint of the interval, with $b_r^2 = (1 - a^2)/2$. Thus the minimum value of $\sigma_x^2 \sigma_y^2$ occurs when $b_r^2 = 0, 1 - a^2$. If $b_r^2 = 0$, b is imaginary, while if $b_r^2 = 1 - a^2$, $b_i = 0$ and b is real. In either case we have

$$(12) \quad \sigma_x \sigma_y = \frac{\hbar^2}{4} |1 - 2a^2| = \frac{\hbar^2}{4} (|b|^2 - a^2) = \frac{\hbar}{2} |\langle S_z \rangle|$$

From the general uncertainty principle, we have

$$(13) \quad \sigma_x^2 \sigma_y^2 \geq \left(\frac{1}{2i} \langle [S_x, S_y] \rangle \right)^2$$

$$(14) \quad = \frac{\hbar^2}{4} |\langle S_z \rangle|^2$$

Thus the minimum uncertainty for the spin 1/2 state is actually the lower limit. We have used absolute values when taking the square root, since uncertainties must be positive.