SPIN 1/2: MINIMUM UNCERTAINTY

For spin 1/2, we can work out the uncertainty relation for the spin components. if we have a general spin 1/2 state, it can be written as

$$\chi = a\chi_+ + b\chi_-$$  \hspace{1cm} (1)

where $\chi_+, -$ are the eigenspinors of $S_z$ and $a$ and $b$ are complex constants. Using the results on spin expectation values, we have

$$\sigma_x^2 = \langle S_x^2 \rangle - \langle S_x \rangle^2$$

$$= \frac{\hbar^2}{4}(1 - (a\ast b - ab\ast)^2)$$  \hspace{1cm} (3)

$$\sigma_y^2 = \langle S_y^2 \rangle - \langle S_y \rangle^2$$

$$= \frac{\hbar^2}{4}(1 - (-a\ast bi + ab\ast i)^2)$$  \hspace{1cm} (5)

The normalization condition

$$|a|^2 + |b|^2 = 1$$  \hspace{1cm} (6)

means that we if $a = a_r e^{i\theta_a}$ and $b = b_r e^{i\theta_b}$ we can transfer the phase of $a$ to $b$ by redefining $a = a_r$ and $b = b_r e^{i(\theta_b - \theta_a)}$ without violating either the relative phase between $a$ and $b$ or the normalization condition.

So, taking $a$ to be real, and $b = b_r + ib_i$
In the fourth line, we used the normalization condition 6.

The term $16a^4b_r^2(1 - a^2 - b_r^2)$ has the form of a parabola such as $Ax(B - x)$ with $A = 16a^4, B = 1 - a^2$ and $x = b_r^2$. Since $b_r^2$ is restricted to the interval $[0, 1 - a^2]$ we see that this portion of a parabola has its minimum values when $b_r^2 = 0, 1 - a^2$ and its maximum value at the midpoint of the interval, with $b_r^2 = (1 - a^2)/2$. Thus the minimum value of $\sigma^2_x \sigma^2_y$ occurs when $b_r^2 = 0, 1 - a^2$. If $b_r^2 = 0$, $b$ is imaginary, while if $b_r^2 = 1 - a^2, b_i = 0$ and $b$ is real. In either case we have

$$\sigma_x \sigma_y = \frac{\hbar^2}{4} |(1 - 2a^2)| = \frac{\hbar^2}{4} |(|b|^2 - a^2)| = \frac{\hbar}{2} |\langle S_z \rangle| \quad (12)$$

From the [general uncertainty principle], we have

$$\sigma^2_x \sigma^2_y \geq \left( \frac{1}{2i} \langle [S_x, S_y] \rangle \right)^2 \quad (13)$$

$$= \frac{\hbar^2}{4} |\langle S_z \rangle|^2 \quad (14)$$

Thus the minimum uncertainty for the spin 1/2 state is actually the lower limit. We have used absolute values when taking the square root, since uncertainties must be positive.