

## EXCHANGE FORCE: INFINITE SQUARE WELL

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Section 5.1.2 & Problem 5.6.

We've seen that distinguishable particles and identical particles must be treated differently in quantum mechanics, resulting in different combinations of the single-particle wave functions in 2-particle systems. It's useful to work out what this means for some of the observables in a 2-particle system.

We can begin by looking at possibly the simplest case: the average positions of the two particles. If the particles are distinguishable, then the wave function is  $\psi(x_a, x_b) = \psi_1(x_a) \psi_2(x_b)$  and

$$\begin{aligned}
 (0.1) \quad \langle x_a \rangle &= \langle \psi | x_a | \psi \rangle \\
 (0.2) \quad &= \langle \psi_{1a} | x_a | \psi_{1a} \rangle \langle \psi_{2b} | \psi_{2b} \rangle \\
 (0.3) \quad &= \langle \psi_{1a} | x_a | \psi_{1a} \rangle \\
 (0.4) \quad &= \langle x \rangle_1
 \end{aligned}$$

where the notation  $|\psi_{1a}\rangle \equiv \psi_1(x_a)$  and so on.

That is,  $\langle x \rangle$  is the mean value of  $x$  in state  $\psi_1$ . We can drop the suffix  $a$  here, since  $x_a$  is just a dummy name for the integration variable in  $\langle \psi_{1a} | x_a | \psi_{1a} \rangle$ .

For identical particles,

$$(0.5) \quad \psi_{\pm}(\mathbf{r}_a, \mathbf{r}_b) = \frac{1}{\sqrt{2}} [\psi_1(\mathbf{r}_a) \psi_2(\mathbf{r}_b) \pm \psi_2(\mathbf{r}_a) \psi_1(\mathbf{r}_b)]$$

This time, working out  $\langle x_a \rangle$  is a bit messier but not too bad if we use the orthogonality of the two states.

$$(0.6) \quad 2 \langle x_a \rangle = \langle \psi_{1a} | x_a | \psi_{1a} \rangle \langle \psi_{2b} | \psi_{2b} \rangle + \langle \psi_{2a} | x_a | \psi_{2a} \rangle \langle \psi_{1b} | \psi_{1b} \rangle$$

$$(0.7) \quad \pm \langle \psi_{1a} | x_a | \psi_{2a} \rangle \langle \psi_{2b} | \psi_{1b} \rangle \pm \langle \psi_{2a} | x_a | \psi_{1a} \rangle \langle \psi_{1b} | \psi_{2b} \rangle$$

$$(0.8) \quad \langle x_a \rangle = \frac{1}{2} (\langle x \rangle_1 + \langle x \rangle_2)$$

Thus the mean position of particle  $a$  is the average of its positions in the two states, which isn't all that surprising. We'd get the same result for

particle  $b$  of course, since the two particles are identical. This result is true for both bosons and fermions, since the plus/minus terms work out to zero due to the orthogonality of the states  $\psi_1$  and  $\psi_2$ .

What is a bit more interesting is the mean square separation of the two particles, that is  $\langle (x_a - x_b)^2 \rangle$ . This can be worked out using the same procedure as above, and is done by Griffiths in his section 5.1.2, although his notation is a bit different from mine. (I've used a numerical suffix on the wave function, since this is the usual notation used for stationary states. Thus a letter suffix indicates which particle and a number suffix indicates which stationary state.) The results are, in my notation, first for distinguishable particles:

$$(0.9) \quad \langle (x_a - x_b)^2 \rangle = \langle x^2 \rangle_1 + \langle x^2 \rangle_2 - 2 \langle x \rangle_1 \langle x \rangle_2$$

For identical particles, we get

$$(0.10) \quad \langle (x_a - x_b)^2 \rangle_{\pm} = \langle x^2 \rangle_1 + \langle x^2 \rangle_2 - 2 \langle x \rangle_1 \langle x \rangle_2 \mp 2 |\langle x \rangle_{12}|^2$$

where the plus (minus) sign on the left and minus (plus) on the right refer to bosons (fermions), and

$$(0.11) \quad \langle x \rangle_{12} \equiv \langle \psi_1 | x | \psi_2 \rangle$$

In general, then, since the term  $2 |\langle x \rangle_{12}|^2$  is always positive, bosons tend to be closer together than distinguishable particles while fermions are further apart. This is a sort of pseudo-force which is an entirely quantum mechanical effect of the symmetries of the wave functions. Although it's not really a force in the sense that electromagnetism and gravity are, it's known as the *exchange force*.

As an example, consider 2 particles in the infinite square well. The wave functions for a single particle are

$$(0.12) \quad \psi(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$$

where  $a$  is the width of the well. If the total wave function is a combination of states  $l$  and  $n$ , then if the particles are distinguishable

(0.13)

$$\langle (x_a - x_b)^2 \rangle = \langle x^2 \rangle_1 + \langle x^2 \rangle_2 - 2 \langle x \rangle_1 \langle x \rangle_2$$

$$(0.14) \quad = a^2 \left( \frac{1}{3} - \frac{1}{2l^2\pi^2} \right) + a^2 \left( \frac{1}{3} - \frac{1}{2n^2\pi^2} \right) - 2 \left( \frac{a}{2} \right) \left( \frac{a}{2} \right)$$

$$(0.15) \quad = a^2 \left( \frac{1}{6} - \frac{l^2 + n^2}{2(\pi ln)^2} \right)$$

In line 2, we used the results of our earlier calculations for the infinite square well.

If the particles are identical, then

$$(0.16) \quad \langle x \rangle_{ln} = \langle \psi_l | x | \psi_n \rangle$$

$$(0.17) \quad = \frac{2}{a} \int_0^a \sin \left( \frac{l\pi x}{a} \right) \sin \left( \frac{n\pi x}{a} \right) x dx$$

$$(0.18) \quad = \left( -1 + (-1)^{n+l} \right) \frac{4anl}{[\pi(n^2 - l^2)]^2}$$

This term is zero if  $n + l$  is even, so there is a difference in the separation only when  $n + l$  is odd. In general, we have

(0.19)

$$\langle (x_a - x_b)^2 \rangle_{\pm} = a^2 \left( \frac{1}{6} - \frac{l^2 + n^2}{2(\pi ln)^2} \right) \mp 2 \left[ \left( -1 + (-1)^{n+l} \right) \frac{4anl}{[\pi(n^2 - l^2)]^2} \right]^2$$

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