EXCHANGE FORCE: INFINITE SQUARE WELL

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Section 5.1.2 & Problem 5.6.

We've seen that distinguishable particles and identical particles must be treated differently in quantum mechanics, resulting in different combinations of the single-particle wave functions in 2-particle systems. It's useful to work out what this means for some of the observables in a 2-particle system.

We can begin by looking at possibly the simplest case: the average positions of the two particles. If the particles are distinguishable, then the wave function is $\psi(x_a,x_b)=\psi_1(x_a)\psi_2(x_b)$ and

$$\langle x_a\rangle = \langle\psi|x_a|\psi\rangle \tag{1}$$
$$= \langle\psi_{1a}|x_a|\psi_{1a}\rangle\langle\psi_{2b}|\psi_{2b}\rangle \tag{2}$$
$$= \langle\psi_{1a}|x_a|\psi_{1a}\rangle \tag{3}$$
$$= \langle x\rangle_1 \tag{4}$$

where the notation $|\psi_{1a}\rangle\equiv\psi_1(x_a)$ and so on.
That is, $\langle x\rangle$ is the mean value of x in state ψ_1. We can drop the suffix a here, since x_a is just a dummy name for the integration variable in $\langle\psi_{1a}|x_a|\psi_{1a}\rangle$.

For identical particles,

$$\psi_\pm(\mathbf{r}_a,\mathbf{r}_b)=\frac{1}{\sqrt{2}}\left[\psi_1(\mathbf{r}_a)\psi_2(\mathbf{r}_b)\pm\psi_2(\mathbf{r}_a)\psi_1(\mathbf{r}_b)\right] \tag{5}$$

This time, working out $\langle x_a\rangle$ is a bit messier but not too bad if we use the orthogonality of the two states.

$$2\langle x_a\rangle = \langle\psi_{1a}|x_a|\psi_{1a}\rangle\langle\psi_{2b}|\psi_{2b}\rangle+\langle\psi_{2a}|x_a|\psi_{2a}\rangle\langle\psi_{1b}|\psi_{1b}\rangle \tag{6}$$
$$\pm\langle\psi_{1a}|x_a|\psi_{2a}\rangle\langle\psi_{2b}|\psi_{1b}\rangle\pm\langle\psi_{2a}|x_a|\psi_{1a}\rangle\langle\psi_{1b}|\psi_{2b}\rangle \tag{7}$$
$$\langle x_a\rangle = \frac{1}{2}\left(\langle x\rangle_1+\langle x\rangle_2\right) \tag{8}$$

Thus the mean position of particle a is the average of its positions in the two states, which isn't all that surprising. We'd get the same result for particle b of course, since the two particles are identical. This result is true for both bosons and fermions, since the plus/minus terms work out to zero due to the orthogonality of the states ψ_1 and ψ_2.

What is a bit more interesting is the mean square separation of the two particles, that is $\left\langle (x_a - x_b)^2 \right\rangle$. This can be worked out using the same procedure as above, and is done by Griffiths in his section 5.1.2, although his notation is a bit different from mine. (I've used a numerical suffix on the wave function, since this is the usual notation used for stationary states. Thus a letter suffix indicates which particle and a number suffix indicates which stationary state.) The results are, in my notation, first for distinguishable particles:

$$\left\langle (x_a - x_b)^2 \right\rangle = \left\langle x^2 \right\rangle_1 + \left\langle x^2 \right\rangle_2 - 2\left\langle x \right\rangle_1 \left\langle x \right\rangle_2 \tag{9}$$

For identical particles, we get

$$\left\langle (x_a - x_b)^2 \right\rangle_{\pm} = \left\langle x^2 \right\rangle_1 + \left\langle x^2 \right\rangle_2 - 2\left\langle x \right\rangle_1 \left\langle x \right\rangle_2 \mp 2\left|\left\langle x \right\rangle_{12}\right|^2 \tag{10}$$

where the plus (minus) sign on the left and minus (plus) on the right refer to bosons (fermions), and

$$\left\langle x \right\rangle_{12} \equiv \left\langle \psi_1 \right| x \left| \psi_2 \right\rangle \tag{11}$$

In general, then, since the term $2\left|\left\langle x \right\rangle_{12}\right|^2$ is always positive, bosons tend to be closer together than distinguishable particles while fermions are further apart. This is a sort of pseudo-force which is an entirely quantum mechanical effect of the symmetries of the wave functions. Although it's not really a force in the sense that electromagnetism and gravity are, it's known as the *exchange force*.

As an example, consider 2 particles in the infinite square well. The wave functions for a single particle are

$$\psi(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \tag{12}$$

where a is the width of the well. If the total wave function is a combination of states l and n, then if the particles are distinguishable

$$\left\langle (x_a - x_b)^2 \right\rangle = \left\langle x^2 \right\rangle_1 + \left\langle x^2 \right\rangle_2 - 2\left\langle x \right\rangle_1 \left\langle x \right\rangle_2 \tag{13}$$

$$= a^2\left(\frac{1}{3} - \frac{1}{2l^2\pi^2}\right) + a^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right) - 2\left(\frac{a}{2}\right)\left(\frac{a}{2}\right) \tag{14}$$

$$= a^2\left(\frac{1}{6} - \frac{l^2+n^2}{2(\pi l n)^2}\right) \tag{15}$$

In line 2, we used the results of our earlier calculations for the infinite square well.

If the particles are identical, then

$$\langle x \rangle_{ln} = \langle \psi_l | x | \psi_n \rangle \tag{16}$$

$$= \frac{2}{a}\int_0^a \sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right) x dx \tag{17}$$

$$= \left(-1 + (-1)^{n+l}\right)\frac{4anl}{\left[\pi\left(n^2 - l^2\right)\right]^2} \tag{18}$$

This term is zero if $n+l$ is even, so there is a difference in the separation only when $n+l$ is odd. In general, we have

$$\left\langle (x_a - x_b)^2 \right\rangle_{\pm} = a^2\left(\frac{1}{6} - \frac{l^2+n^2}{2(\pi l n)^2}\right) \mp 2\left[\left(-1 + (-1)^{n+l}\right)\frac{4anl}{\left[\pi\left(n^2 - l^2\right)\right]^2}\right]^2 \tag{19}$$

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