

EXCHANGE FORCE: INFINITE SQUARE WELL

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Section 5.1.2 & Problem 5.6.

We've seen that distinguishable particles and identical particles must be treated differently in quantum mechanics, resulting in different combinations of the single-particle wave functions in 2-particle systems. It's useful to work out what this means for some of the observables in a 2-particle system.

We can begin by looking at possibly the simplest case: the average positions of the two particles. If the particles are distinguishable, then the wave function is $\psi(x_a, x_b) = \psi_1(x_a) \psi_2(x_b)$ and

$$\langle x_a \rangle = \langle \psi | x_a | \psi \rangle \quad (1)$$

$$= \langle \psi_{1a} | x_a | \psi_{1a} \rangle \langle \psi_{2b} | \psi_{2b} \rangle \quad (2)$$

$$= \langle \psi_{1a} | x_a | \psi_{1a} \rangle \quad (3)$$

$$= \langle x \rangle_1 \quad (4)$$

where the notation $|\psi_{1a}\rangle \equiv \psi_1(x_a)$ and so on.

That is, $\langle x \rangle$ is the mean value of x in state ψ_1 . We can drop the suffix a here, since x_a is just a dummy name for the integration variable in $\langle \psi_{1a} | x_a | \psi_{1a} \rangle$.

For identical particles,

$$\psi_{\pm}(\mathbf{r}_a, \mathbf{r}_b) = \frac{1}{\sqrt{2}} [\psi_1(\mathbf{r}_a) \psi_2(\mathbf{r}_b) \pm \psi_2(\mathbf{r}_a) \psi_1(\mathbf{r}_b)] \quad (5)$$

This time, working out $\langle x_a \rangle$ is a bit messier but not too bad if we use the orthogonality of the two states.

$$2 \langle x_a \rangle = \langle \psi_{1a} | x_a | \psi_{1a} \rangle \langle \psi_{2b} | \psi_{2b} \rangle + \langle \psi_{2a} | x_a | \psi_{2a} \rangle \langle \psi_{1b} | \psi_{1b} \rangle \quad (6)$$

$$\pm \langle \psi_{1a} | x_a | \psi_{2a} \rangle \langle \psi_{2b} | \psi_{1b} \rangle \pm \langle \psi_{2a} | x_a | \psi_{1a} \rangle \langle \psi_{1b} | \psi_{2b} \rangle \quad (7)$$

$$\langle x_a \rangle = \frac{1}{2} (\langle x \rangle_1 + \langle x \rangle_2) \quad (8)$$

Thus the mean position of particle a is the average of its positions in the two states, which isn't all that surprising. We'd get the same result for

particle b of course, since the two particles are identical. This result is true for both bosons and fermions, since the plus/minus terms work out to zero due to the orthogonality of the states ψ_1 and ψ_2 .

What is a bit more interesting is the mean square separation of the two particles, that is $\langle (x_a - x_b)^2 \rangle$. This can be worked out using the same procedure as above, and is done by Griffiths in his section 5.1.2, although his notation is a bit different from mine. (I've used a numerical suffix on the wave function, since this is the usual notation used for stationary states. Thus a letter suffix indicates which particle and a number suffix indicates which stationary state.) The results are, in my notation, first for distinguishable particles:

$$\langle (x_a - x_b)^2 \rangle = \langle x^2 \rangle_1 + \langle x^2 \rangle_2 - 2 \langle x \rangle_1 \langle x \rangle_2 \quad (9)$$

For identical particles, we get

$$\langle (x_a - x_b)^2 \rangle_{\pm} = \langle x^2 \rangle_1 + \langle x^2 \rangle_2 - 2 \langle x \rangle_1 \langle x \rangle_2 \mp 2 |\langle x \rangle_{12}|^2 \quad (10)$$

where the plus (minus) sign on the left and minus (plus) on the right refer to bosons (fermions), and

$$\langle x \rangle_{12} \equiv \langle \psi_1 | x | \psi_2 \rangle \quad (11)$$

In general, then, since the term $2 |\langle x \rangle_{12}|^2$ is always positive, bosons tend to be closer together than distinguishable particles while fermions are further apart. This is a sort of pseudo-force which is an entirely quantum mechanical effect of the symmetries of the wave functions. Although it's not really a force in the sense that electromagnetism and gravity are, it's known as the *exchange force*.

As an example, consider 2 particles in the infinite square well. The wave functions for a single particle are

$$\psi(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \quad (12)$$

where a is the width of the well. If the total wave function is a combination of states l and n , then if the particles are distinguishable

$$\langle (x_a - x_b)^2 \rangle = \langle x^2 \rangle_1 + \langle x^2 \rangle_2 - 2 \langle x \rangle_1 \langle x \rangle_2 \quad (13)$$

$$= a^2 \left(\frac{1}{3} - \frac{1}{2l^2\pi^2} \right) + a^2 \left(\frac{1}{3} - \frac{1}{2n^2\pi^2} \right) - 2 \left(\frac{a}{2} \right) \left(\frac{a}{2} \right) \quad (14)$$

$$= a^2 \left(\frac{1}{6} - \frac{l^2 + n^2}{2(\pi ln)^2} \right) \quad (15)$$

In line 2, we used the results of our earlier calculations for the infinite square well.

If the particles are identical, then

$$\langle x \rangle_{ln} = \langle \psi_l | x | \psi_n \rangle \quad (16)$$

$$= \frac{2}{a} \int_0^a \sin \left(\frac{l\pi x}{a} \right) \sin \left(\frac{n\pi x}{a} \right) x dx \quad (17)$$

$$= \left(-1 + (-1)^{n+l} \right) \frac{4anl}{[\pi(n^2 - l^2)]^2} \quad (18)$$

This term is zero if $n + l$ is even, so there is a difference in the separation only when $n + l$ is odd. In general, we have

$$\langle (x_a - x_b)^2 \rangle_{\pm} = a^2 \left(\frac{1}{6} - \frac{l^2 + n^2}{2(\pi ln)^2} \right) \mp 2 \left[\left(-1 + (-1)^{n+l} \right) \frac{4anl}{[\pi(n^2 - l^2)]^2} \right]^2 \quad (19)$$

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