STATISTICAL MECHANICS IN QUANTUM THEORY: ENERGY PROBABILITIES

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 5.23.

In quantum statistical mechanics, we can work out the probability that a particle chosen at random has a given energy by working out a few basic probabilities. The central assumption is that all states with a given energy are equally probable, if the system is in thermal equilibrium, which means that it is not exchanging energy with its surroundings. For 3 particles in the simple harmonic oscillator potential with a total energy of $E=\frac{9}{2}\hbar\omega$, we first need to work out the possible state combinations. Since the energy levels of the harmonic oscillator are $E_n=\left(n+\frac{1}{2}\right)\hbar\omega$, we can list the combinations that total to $\frac{9}{2}\hbar\omega$, where we list the values of n for each particle:

$$(1,1,1)1\text{ state} \tag{1}$$

$$(0,0,3)3\text{ states} \tag{2}$$

$$(0,1,2)6\text{ states} \tag{3}$$

For distinguishable particles, all 10 states are possible, so the probabilities of a particle having a given energy are:

$$E=\frac{\hbar\omega}{2};\;\; p\left(\frac{\hbar\omega}{2}\right)=\frac{2}{3}p(0,0,3)+\frac{1}{3}p(0,1,2)=\frac{2}{3}\times0.3+\frac{1}{3}\times0.6=0.4 \tag{4}$$

$$E=\frac{3\hbar\omega}{2};\;\; p\left(\frac{3\hbar\omega}{2}\right)=p(1,1,1)+\frac{1}{3}p(0,1,2)=0.3 \tag{5}$$

$$E=\frac{5\hbar\omega}{2};\;\; p\left(\frac{5\hbar\omega}{2}\right)=\frac{1}{3}p(0,1,2)=0.2 \tag{6}$$

$$E=\frac{7\hbar\omega}{2};\;\; p\left(\frac{7\hbar\omega}{2}\right)=\frac{1}{3}p(0,0,3)=0.1 \tag{7}$$

If the particles are identical fermions each fermion must be in a different state, so only one overall state is possible, which is the antisymmetric combination of the 6 permutations of (0,1,2). A particle has an equal chance of being in any of the three energy states so

$$p\left(\frac{\hbar\omega}{2}\right)=p\left(\frac{3\hbar\omega}{2}\right)=p\left(\frac{5\hbar\omega}{2}\right)=\frac{1}{3} \tag{8}$$

For identical bosons, the total state could be (1,1,1) or the symmetric combination of the 3 (0,0,3) states, or the symmetric combination of the 6 (0,1,2) states. Each of these 3 overall states is equally probable, so the probabilities for an individual particle are

$$p\left(\frac{\hbar\omega}{2}\right)=\frac{1}{3}\left(\frac{2}{3}+\frac{1}{3}\right)=\frac{1}{3} \tag{9}$$

$$p\left(\frac{3\hbar\omega}{2}\right)=\frac{1}{3}\left(1+\frac{1}{3}\right)=\frac{4}{9} \tag{10}$$

$$p\left(\frac{5\hbar\omega}{2}\right)=\frac{1}{3}\times\frac{1}{3}=\frac{1}{9} \tag{11}$$

$$p\left(\frac{7\hbar\omega}{2}\right)=\frac{1}{3}\times\frac{1}{3}=\frac{1}{9} \tag{12}$$