

STATISTICAL MECHANICS IN QUANTUM THEORY: ENERGY PROBABILITIES

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 5.23.

In quantum statistical mechanics, we can work out the probability that a particle chosen at random has a given energy by working out a few basic probabilities. The central assumption is that all states with a given energy are equally probable, if the system is in thermal equilibrium, which means that it is not exchanging energy with its surroundings. For 3 particles in the simple harmonic oscillator potential with a total energy of $E = \frac{9}{2}\hbar\omega$, we first need to work out the possible state combinations. Since the energy levels of the harmonic oscillator are $E_n = (n + \frac{1}{2})\hbar\omega$, we can list the combinations that total to $\frac{9}{2}\hbar\omega$, where we list the values of n for each particle:

$$(1, 1, 1) 1 \text{ state} \quad (1)$$

$$(0, 0, 3) 3 \text{ states} \quad (2)$$

$$(0, 1, 2) 6 \text{ states} \quad (3)$$

For distinguishable particles, all 10 states are possible, so the probabilities of a particle having a given energy are:

$$E = \frac{\hbar\omega}{2}; \quad p\left(\frac{\hbar\omega}{2}\right) = \frac{2}{3}p(0,0,3) + \frac{1}{3}p(0,1,2) = \frac{2}{3} \times 0.3 + \frac{1}{3} \times 0.6 = 0.4 \quad (4)$$

$$E = \frac{3\hbar\omega}{2}; \quad p\left(\frac{3\hbar\omega}{2}\right) = p(1,1,1) + \frac{1}{3}p(0,1,2) = 0.3 \quad (5)$$

$$E = \frac{5\hbar\omega}{2}; \quad p\left(\frac{5\hbar\omega}{2}\right) = \frac{1}{3}p(0,1,2) = 0.2 \quad (6)$$

$$E = \frac{7\hbar\omega}{2}; \quad p\left(\frac{7\hbar\omega}{2}\right) = \frac{1}{3}p(0,0,3) = 0.1 \quad (7)$$

If the particles are identical fermions each fermion must be in a different state, so only one overall state is possible, which is the antisymmetric combination of the 6 permutations of (0,1,2). A particle has an equal chance of being in any of the three energy states so

$$p\left(\frac{\hbar\omega}{2}\right) = p\left(\frac{3\hbar\omega}{2}\right) = p\left(\frac{5\hbar\omega}{2}\right) = \frac{1}{3} \quad (8)$$

For identical bosons, the total state could be (1,1,1) or the symmetric combination of the 3 (0,0,3) states, or the symmetric combination of the 6 (0,1,2) states. Each of these 3 overall states is equally probable, so the probabilities for an individual particle are

$$p\left(\frac{\hbar\omega}{2}\right) = \frac{1}{3} \left(\frac{2}{3} + \frac{1}{3} \right) = \frac{1}{3} \quad (9)$$

$$p\left(\frac{3\hbar\omega}{2}\right) = \frac{1}{3} \left(1 + \frac{1}{3} \right) = \frac{4}{9} \quad (10)$$

$$p\left(\frac{5\hbar\omega}{2}\right) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9} \quad (11)$$

$$p\left(\frac{7\hbar\omega}{2}\right) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9} \quad (12)$$