STATISTICAL MECHANICS IN QUANTUM THEORY: COUNTING STATES, GENERAL CASE

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 5.24.

Our first look at counting states available to particles assumed that each energy state was unique. In many potentials, energy states are degenerate so we need to take this into account when calculating the number of available states. As usual, the calculations depend on the type of particle (distinguishable, fermions or bosons) so we need to look at each case separately.

First, we'll look at distinguishable particles. In that case, we've seen that for non-degenerate energy states, the total number of ways we can distribute N particles among m states is

$$S_d(\{n_j\}) = N!\prod_{j=1}^{m}\frac{1}{n_j!} \tag{1}$$

where $\sum_{j=1}^{m} n_j = N$. If state j has a degeneracy of d_j, then each particle in that state has a choice of d_j substates it can occupy, so this multiplies the number of available states by a factor of $d_j^{n_j}$ for each state j. Thus the formula above is modified to

$$S_d(\{n_j\}) = N!\prod_{j=1}^{m}\frac{d_j^{n_j}}{n_j!} \tag{2}$$

For fermions, since all particles must occupy different states, there can be at most d_j particles in energy level j. Thus if there are n_j particles in level j, the number of ways they can fit into that level is $\binom{d_j}{n_j}$. The total number of states is then

$$S_f(\{n_j\}) = \prod_{j=1}^{m}\binom{d_j}{n_j} \tag{3}$$

This formula actually does work even if $n_j > d_j$, since then there will be a factor of $(d_j - n_j)!$ in the denominator, and factorials of negative integers (actually a special case of the gamma function) are infinite, so the overall product becomes zero meaning that it is impossible to put more than d_j particles into level j. In the example of 3 particles we looked at earlier,

$d_j = 1$ for all states. For states such as (11,11,11), (5,13,13) and (1,1,19) one of the n_j is greater than 1, meaning that the corresponding $S_f = 0$. For the state (5,7,17), all the $n_j = 1$, so $S_f = 1$ meaning that there is only one state available (which is the antisymmetrized combination of the 6 permutations of the state (5,7,17)). Another way of looking at it is, of course, that unless all the single-particle states are different, it's impossible to form a fully antisymmetrized combination of the individual states.

For bosons, we want to place n_j *identical* particles into a state with a degeneracy of d_j. In this case, we're allowed to have more than one particle in a given state, so it's possible that $n_j > d_j$. Since the particles are identical, all we are concerned with is *how many* (rather than *which*) particles are in each degenerate substate. One way of working out the number of combinations is to imagine n_j particles and $d_j - 1$ dividers arranged in a line. If all these objects were distinct, there are $(n_j + d_j - 1)!$ ways of arranging them. However, because the n_j particles are identical, as are the $d_j - 1$ dividers, their orders don't matter, and the total number of ways of arranging the particles and dividers is $\binom{n_j+d_j-1}{n_j}$. Thus the total number of ways of placing N particles in m states is

$$S_b(\{n_j\}) = \prod_{j=1}^{m} \binom{n_j + d_j - 1}{n_j} \tag{4}$$

In the case where all $d_j = 1$ (as in our earlier example), then

$$S_b(\{n_j\}) = \prod_{j=1}^{m} \binom{n_j}{n_j} = 1 \tag{5}$$

This is because, for any given energy state, the compound boson state must be a symmetrized combination of the individual particle states, and there is only one way of doing this for non-degenerate states.

PINGBACKS

Pingback: Statistical mechanics in quantum theory: counting boson states
Pingback: Statistical mechanics in quantum theory: most probable state