STATISTICAL MECHANICS IN QUANTUM THEORY: COUNTING BOSON STATES

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 5.25.

When working out the number of possible configurations in which N identical bosons can be placed into a set of energy states, each of which can be degenerate, we came to the intermediate result of the number of ways of distributing n bosons among the d degenerate levels in a single energy state:

$$S_b(n) = \binom{n+d-1}{n} \tag{1}$$

Although we proved this by a simple combinatorial argument, it is also possible to prove this formula by a slightly more involved route. We start with the lowest values of n and attempt to deduce a pattern.

With $n=1$, there are d ways of assigning them in d bins, so

$$S_b(1) = d = \binom{d}{1} \tag{2}$$

With $n=2$, we could put both bosons in the same bin ($d=\binom{d}{1}$ ways) or one in each of two bins $\binom{d}{2}$ ways. Thus the total is

$$S_b(2) = \binom{d}{1} + \binom{d}{2} \tag{3}$$

With $n=3$, we could put all 3 in the same bin ($d=\binom{d}{1}$ ways), or 2 in one bin and 1 in another ($2\binom{d}{2}$ ways, since if we choose 2 bins, swapping the bin that contains 2 with the bin that contains 1 is a different configuration), or all 3 in separate bins ($\binom{d}{3}$ ways). The total is then

$$S_b(3) = \binom{d}{1} + 2\binom{d}{2} + \binom{d}{3} \tag{4}$$

For $n=4$, we can have all 4 in the same bin ($\binom{d}{1}$ ways), 3 in one bin and 1 in another ($2\binom{d}{2}$ ways), 2 in one bin and 2 in another ($\binom{d}{2}$ ways), 2 in one bin and the other 2 in separate bins ($3\binom{d}{3}$ ways) or all 4 in separate bins ($\binom{d}{4}$ ways) making a total of

$$S_b(4)=\binom{d}{1}+3\binom{d}{2}+3\binom{d}{3}+\binom{d}{4} \tag{5}$$

The coefficients of each term on the RHS appear to be the binomial coefficients $\binom{n-1}{j-1}$ for $j=1,\ldots,n$. That is, the general form appears to be

$$S_b(n)=\sum_{j=1}^{n}\binom{d}{j}\binom{n-1}{j-1} \tag{6}$$

We can verify this by the following argument. First we need to work out the number of ways of dividing n bosons into j groups. We can do this by arranging the bosons in a line and placing a partition between each pair of particles, so there are $n-1$ partitions. To divide the bosons into j groups, we select $j-1$ of these partitions and ignore the rest. The number of ways we can select the partitions is then $\binom{n-1}{j-1}$. Once we have selected the groups of bosons, we need to distribute them among the d degenerate states, and this can be done in $\binom{d}{j}$ ways. Combining these two gives the result above.

It might seem that something is a bit wonky with this argument. The formula for the number of ways of partitioning n bosons assumes, for example, that if we had 3 bosons, putting the partition between bosons 1 and 2 is different from putting it between 2 and 3, whereas both partitions divide the bosons into a group of 2 and a group of 1. Since bosons are identical particles, these two groupings should be the same, so it looks like we're overcounting.

The point is that the other factor of $\binom{d}{j}$ *undercounts*, since it assumes that the order in which the members of the partition are assigned to degenerate states doesn't matter, whereas because all the degenerate states are different, the order *does* matter. Thus retaining the order in the original partitioning compensates for neglecting it when assigning groups to degenerate states, and the total result is in fact correct.

By using the symmetry of the binomial coefficient:

$$\binom{n-1}{j-1}=\binom{n-1}{n-j} \tag{7}$$

we can use Vandermonde's identity to say that

$$\begin{aligned} S_b(n) &= \sum_{j=1}^{n}\binom{d}{j}\binom{n-1}{n-j} && (8)\\ &= \binom{d+n-1}{n} && (9)\end{aligned}$$

As far as I know, there is no induction proof of Vandermonde's identity, but for a couple of other proofs, see the link above.