BLACKBODY RADIATION

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 5.30.

The photon is a special type of boson, since it has spin 1. However, it is (pretty much by definition) a relativistic particle, since it has zero mass and always travels at the speed of light c. As such, we really need relativistic quantum theory to analyze photons properly, but we can use the formula for the number distribution of bosons if we incorporate some results from the relativistic theory (necessarily on faith for now - eventually I hope to study quantum field theory).

The number distribution for bosons that we derived earlier is

$$n_j = \frac{d_j}{e^{\alpha+\beta E_j} - 1} \tag{0.1}$$

where α and β are the Lagrange multipliers that arise from the constraint of constant particle number and constant total energy, respectively. One of the peculiarities of photons is that their number is not necessarily conserved. Since a photon's speed is fixed at c and its energy is determined by its frequency $E = h\nu = \hbar\omega$, a photon cannot change its energy as the temperature changes. A system reacts to a change in temperature by changing the number of photons present. We can drop the constraint of constant number by setting $\alpha = 0$, so the number distribution becomes (using $\beta = 1/k_BT$ and $E = \hbar\omega$):

$$n_\omega = \frac{d_k}{e^{\hbar\omega/k_BT} - 1} \tag{0.2}$$

To calculate the degeneracy d_k we have to invoke a couple of other assumptions. First, we'll assume that we can use the same calculation as in the non-relativistic case to work out d_k. That is, the number of states in the shell from k to $k + dk$ is

$$d_k = \frac{k^2 V}{2\pi^2} dk \tag{0.3}$$

The second assumption is that $k = \frac{2\pi}{\lambda} = \frac{\omega}{c}$. We can then change variables from k to ω by using $dk = d\omega/c$ so

$$d_\omega = \frac{\omega^2 V}{2\pi^2 c^3} d\omega \tag{0.4}$$

To get the full degeneracy, we need to multiply by the number of available spin states. For spin 1, we might think that the number of states is 3 $(+1, 0, -1)$, however photons are peculiar in that they do not exist in the spin 0 state, so there are only 2 spin states. Thus the total degeneracy, including spin, is

$$d_\omega = \frac{\omega^2 V}{\pi^2 c^3} d\omega \tag{0.5}$$

The energy density is then $($number of photons with frequency $\omega)$ $($energy of 1 photon with frequenc
which is

$$\frac{n_\omega \hbar\omega}{V} = \rho(\omega)\, d\omega \tag{0.6}$$

$$= \frac{1}{\left(e^{\hbar\omega/k_B T} - 1\right)} \frac{\hbar\omega^3}{\pi^2 c^3} d\omega \tag{0.7}$$

This is Planck's formula for the radiation of a black body (a body in thermal equilibrium with a surrounding electromagnetic field). It is a famous formula because it solved the problem of the *ultraviolet catastrophe*, which resulted from the prediction of the Rayleigh-Jeans law of classical physics, which stated that the radiation should follow the distribution $\omega^2 k_B T/\pi^2 c^3$, which says that the radiation should increase quadratically with frequency for all temperatures. This is clearly wrong, as it predicts an infinite amount of energy is emitted at the high-frequency (ultraviolet) end of the spectrum, however it did agree with experiment for low frequencies. It can be seen that Planck's formula reduces to the Rayleigh-Jeans law for small ω, since $e^{\hbar\omega/k_B T} \approx 1 + \frac{\hbar\omega}{k_B T}$ in that limit. However, for high frequencies, Planck's formula return back to zero because of the exponential in the denominator.

We can express the Planck formula in terms of wavelength by noting that $\omega = 2\pi c/\lambda$ so

$$d\omega = -\frac{2\pi c}{\lambda^2} d\lambda \tag{0.8}$$

From Planck's formula:

$$\frac{1}{\left(e^{\hbar\omega/k_BT}-1\right)}\frac{\hbar\omega^3}{\pi^2c^3}d\omega = -\frac{1}{\left(e^{2\pi\hbar c/\lambda k_BT}-1\right)}\frac{\hbar}{\pi^2c^3}\left(\frac{2\pi c}{\lambda}\right)^3\frac{2\pi c}{\lambda^2}d\lambda \tag{0.9}$$

$$= -\frac{1}{\left(e^{2\pi\hbar c/\lambda k_BT}-1\right)}\frac{16\pi^2c\hbar}{\lambda^5}d\lambda \tag{0.10}$$

Thus the energy density (per unit volume and unit wavelength interval) in terms of wavelength is

$$\overline{\rho}(\lambda) = \frac{16\pi^2c\hbar}{\lambda^5}\frac{1}{e^{A/\lambda}-1} \tag{0.11}$$

where

$$A \equiv \frac{2\pi c\hbar}{k_BT} \tag{0.12}$$

The wavelength at which the radiation is a maximum can be found by taking the derivative of $\overline{\rho}(\lambda)$ and setting to zero. We get

$$-\frac{5}{\lambda^6}\frac{1}{e^{A/\lambda}-1}+\frac{1}{\lambda^5}\frac{A}{\lambda^2}\frac{e^{A/\lambda}}{\left(e^{A/\lambda}-1\right)^2} = 0 \tag{0.13}$$

$$5\lambda = A\frac{e^{A/\lambda}}{e^{A/\lambda}-1} \tag{0.14}$$

$$5\frac{\lambda}{A} = \frac{1}{1-e^{-A/\lambda}} \tag{0.15}$$

$$5-\frac{A}{\lambda} = 5e^{-A/\lambda} \tag{0.16}$$

We therefore need to solve the equation numerically. If we define $x \equiv A/\lambda$ the equation is

$$5-x = 5e^{-x} \tag{0.17}$$

which (using Maple) has solutions at $x=0$ and $x=4.965$. Plugging in the constants in the definition of A, we get

$$\lambda_{max} = \frac{A}{4.965} \tag{0.18}$$

$$= \frac{2.901 \times 10^{-3}}{T} \text{ m K} \tag{0.19}$$

This is known as *Wien's displacement law*. It's called a 'displacement' law since it states that the shape of the radiation curve is essentially the same shape at all temperatures, with the maximum wavelength displaced up or down, depending on the temperature.

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