ELECTRON PRESSURE IN A WHITE DWARF STAR

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 5.35.

An exotic application of the electron gas model of matter is that of a white dwarf star, which is composed of a collapsed form of matter that is prevented from collapsing even further (into a neutron star or black hole) by the degeneracy pressure. We can actually use the electron gas model to get an idea of the radius of a white dwarf star.

The total energy of the electrons in the star is

$$E_e=\frac{\hbar^2\left(3\pi^2Nq\right)^{5/3}}{10\pi^2m}V^{-2/3} \tag{0.1}$$

where N is the number of nucleons (protons and neutrons), q is the number of electrons per nucleon and V is the volume of the star. Taking the star to be a sphere, we have $V=\frac{4}{3}\pi R^3$ so

$$E_e=\frac{\hbar^2}{10mR^2}\left(\frac{3\pi}{4}\right)^{2/3}(3Nq)^{5/3} \tag{0.2}$$

To get the total energy in the star we need to include the gravitational potential energy. If we take the star to be of uniform density ρ, then the potential energy of a shell of mass of radius r is $-GM(r)\left(4\pi r^2\rho dr\right)/r$, where $M(r)$ is the mass of the star interior to radius r, so $M(r)=\frac{4}{3}\pi r^3\rho$. Adding up all the shells, we get

$$U=-G\int_0^R\frac{\left(\frac{4}{3}\pi r^3\rho\right)\left(4\pi r^2\rho\right)}{r}dr \tag{0.3}$$

$$=-\frac{16}{15}G(\pi\rho)^2R^5 \tag{0.4}$$

The density can be written as

$$\rho=\frac{Nm_n}{V}=\frac{3Nm_n}{4\pi R^3} \tag{0.5}$$

where m_n is the mass of a nucleon. Substituting this gives us

$$U = -\frac{3}{5R} G N^2 m_n^2 \tag{0.6}$$

The total energy is then

$$E = \frac{\hbar^2}{10mR^2}\left(\frac{3\pi}{4}\right)^{2/3}(3Nq)^{5/3} - \frac{3}{5R} G N^2 m_n^2 \tag{0.7}$$

The star should be in its minimum energy state (where the electron pressure balances the gravitational potential energy), so we can find the minimum of E by taking the derivative with respect to R and setting to zero. This gives a linear equation for R (after multiplying through by R^3), so solving for R we get

$$R = \left(\frac{9\pi}{4}\right)^{2/3} \frac{\hbar^2 q^{5/3}}{G m m_n^2 N^{1/3}} \tag{0.8}$$

Taking $q = 1/2$ (this assumes a roughly equal number of protons and neutrons, with one electron per proton) and plugging in the other constants, we get

$$R = 7.587 \times 10^{25} N^{-1/3} \text{ m} \tag{0.9}$$

If the star has the mass of the sun (1.98892×10^{30} kg), the number of nucleons is

$$N = \frac{1.98892 \times 10^{30}}{1.67272 \times 10^{-27}} = 1.189 \times 10^{57} \tag{0.10}$$

giving a radius of

$$R = 7.162 \times 10^{6} \text{ m} = 7162 \text{ km} \tag{0.11}$$

which is not much larger than the Earth (radius 6378 km).

If the star is in its ground state (which it isn't, but this calculation gives a lowest maximum energy), then the highest energy electrons will have an energy equal to the Fermi energy:

$$E_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 Nq}{V}\right)^{2/3} \tag{0.12}$$

$$= \frac{\hbar^2}{2m}\left(\frac{9\pi Nq}{4R^3}\right)^{2/3} \tag{0.13}$$

$$= 0.1934 \text{ MeV} \tag{0.14}$$

The rest energy of an electron is 0.511 MeV, so the Fermi energy is a sizeable fraction of the rest energy, making their speeds verging on the relativistic zone.

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