

ELECTRON PRESSURE IN A WHITE DWARF STAR

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 5.35.

An exotic application of the electron gas model of matter is that of a white dwarf star, which is composed of a collapsed form of matter that is prevented from collapsing even further (into a neutron star or black hole) by the degeneracy pressure. We can actually use the electron gas model to get an idea of the radius of a white dwarf star.

The total energy of the electrons in the star is

$$E_e = \frac{\hbar^2 (3\pi^2 Nq)^{5/3}}{10\pi^2 m} V^{-2/3} \quad (1)$$

where N is the number of nucleons (protons and neutrons), q is the number of electrons per nucleon and V is the volume of the star. Taking the star to be a sphere, we have $V = \frac{4}{3}\pi R^3$ so

$$E_e = \frac{\hbar^2}{10mR^2} \left(\frac{3\pi}{4}\right)^{2/3} (3Nq)^{5/3} \quad (2)$$

To get the total energy in the star we need to include the gravitational potential energy. If we take the star to be of uniform density ρ , then the potential energy of a shell of mass of radius r is $-GM(r)(4\pi r^2 \rho dr)/r$, where $M(r)$ is the mass of the star interior to radius r , so $M(r) = \frac{4}{3}\pi r^3 \rho$. Adding up all the shells, we get

$$U = -G \int_0^R \frac{(\frac{4}{3}\pi r^3 \rho)(4\pi r^2 \rho)}{r} dr \quad (3)$$

$$= -\frac{16}{15}G(\pi\rho)^2 R^5 \quad (4)$$

The density can be written as

$$\rho = \frac{Nm_n}{V} = \frac{3Nm_n}{4\pi R^3} \quad (5)$$

where m_n is the mass of a nucleon. Substituting this gives us

$$U = -\frac{3}{5R}GN^2m_n^2 \quad (6)$$

The total energy is then

$$E = \frac{\hbar^2}{10mR^2} \left(\frac{3\pi}{4} \right)^{2/3} (3Nq)^{5/3} - \frac{3}{5R}GN^2m_n^2 \quad (7)$$

The star should be in its minimum energy state (where the electron pressure balances the gravitational potential energy), so we can find the minimum of E by taking the derivative with respect to R and setting to zero. This gives a linear equation for R (after multiplying through by R^3), so solving for R we get

$$R = \left(\frac{9\pi}{4} \right)^{2/3} \frac{\hbar^2 q^{5/3}}{Gmm_n^2 N^{1/3}} \quad (8)$$

Taking $q = 1/2$ (this assumes a roughly equal number of protons and neutrons, with one electron per proton) and plugging in the other constants, we get

$$R = 7.587 \times 10^{25} N^{-1/3} \text{ m} \quad (9)$$

If the star has the mass of the sun (1.98892×10^{30} kg), the number of nucleons is

$$N = \frac{1.98892 \times 10^{30}}{1.67272 \times 10^{-27}} = 1.189 \times 10^{57} \quad (10)$$

giving a radius of

$$R = 7.162 \times 10^6 \text{ m} = 7162 \text{ km} \quad (11)$$

which is not much larger than the Earth (radius 6378 km).

If the star is in its ground state (which it isn't, but this calculation gives a lowest maximum energy), then the highest energy electrons will have an energy equal to the Fermi energy:

$$E_F = \frac{\hbar^2}{2m} \left(\frac{3\pi^2 Nq}{V} \right)^{2/3} \quad (12)$$

$$= \frac{\hbar^2}{2m} \left(\frac{9\pi Nq}{4R^3} \right)^{2/3} \quad (13)$$

$$= 0.1934 \text{ MeV} \quad (14)$$

The rest energy of an electron is 0.511 MeV, so the Fermi energy is a sizeable fraction of the rest energy, making their speeds verging on the relativistic zone.

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