ELECTRON PRESSURE IN A NEUTRON STAR

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 5.36.

We can get a rough idea of how electrons fill the available states in the relativistic regime by using the relativistic kinetic energy

$$E=\sqrt{p^2c^2+m^2c^4}-mc^2 \tag{0.1}$$

in place of the Newtonian formula $E=p^2/2m$. This presumably isn't a true relativistic quantum theory, since the rest of the calculations are still based on the Schrödinger equation, but it's interesting to see where it leads.

If we take the momentum to be $\mathbf{p}=\hbar\mathbf{k}$, then in the extreme relativistic case, $pc\gg mc^2$ so $E\approx pc=\hbar ck$. If we use this energy in our previous calculation, the energy of the states in a shell in the first octant in k-space is

$$dE = 2\hbar ck\frac{V}{\pi^3}\left(\frac{1}{8}4\pi k^2\right)dk \tag{0.2}$$

$$= \hbar c\frac{V}{\pi^2}k^3dk \tag{0.3}$$

where the first factor of 2 accounts for the spin degeneracy, V/π^3 is the volume of one state in k-space and the last factor is the volume of a spherical shell in the first octant.

The total energy is then

$$E_{tot} = \hbar c\frac{V}{\pi^2}\int_0^{k_F}k^3dk \tag{0.4}$$

$$= \frac{\hbar ck_F^4V}{4\pi^2} \tag{0.5}$$

where k_F is the Fermi radius (the maximum value of k in the ground state). This is the same as in the previous calculation, so we have

$$k_F = \left(\frac{3\pi^2 Nq}{V}\right)^{1/3} \tag{0.6}$$

$$E_{tot} = \frac{\hbar c}{4\pi^2 V^{1/3}} \left(3\pi^2 Nq\right)^{4/3} \tag{0.7}$$

Using $V = 4\pi R^3/3$ for a sphere, we get

$$E_{tot} = \frac{\hbar c}{4\pi^2} \left(3\pi^2 Nq\right)^{4/3} \left(\frac{3}{4\pi}\right)^{1/3} \frac{1}{R} \tag{0.8}$$

We can now apply this to the case of a star as we did with the white dwarf. The gravitational potential energy of the star is the same:

$$U = -\frac{3}{5R} G N^2 m_n^2 \tag{0.9}$$

where G is the gravitational constant and m_n is the mass of a nucleon.

At this point, we would like to find the minimum of $E_{tot} + U$, but unlike the non-relativistic case, both terms now have a $1/R$ dependence, so there is no minimum. If $E_{tot} + U > 0$, the electron pressure is greater than the gravitational force and the star will expand; in the opposite case, gravity wins out and the star collapses. The critical point occurs when $E_{tot} + U = 0$, that is when

$$E = \left[\frac{\hbar c}{4\pi^2} \left(3\pi^2 Nq\right)^{4/3} \left(\frac{3}{4\pi}\right)^{1/3} - \frac{3}{5} G N^2 m_n^2\right] \frac{1}{R} = 0 \tag{0.10}$$

This doesn't give us a condition on R, but the only variable quantity within the brackets is the number of nucelons N, which can be translated into the mass of the star. The condition is

$$\frac{\hbar c}{4\pi^2} \left(3\pi^2 N_c q\right)^{4/3} \left(\frac{3}{4\pi}\right)^{1/3} = \frac{3}{5} G N_c^2 m_n^2 \tag{0.11}$$

$$N_c = \frac{15\sqrt{5\pi}\,(\hbar c)^{3/2} q^2}{16 G^{3/2} m_n^3} \tag{0.12}$$

$$= 2.047 \times 10^{57} \tag{0.13}$$

(This doesn't agree with the answer given in Griffiths's question, but I think he's made a mistake, since this value gives a Chandrasekhar limit that is closer to the accepted value of 1.4 solar masses.)

Given the solar mass of 1.98892×10^{30} kg and a proton mass of 1.6726×10^{-27} kg, this number of nucleons gives a critical mass of around 1.72 solar masses. This stellar mass is known as the *Chandrasekhar limit*, since stars more massive than this will collapse beyond the white dwarf stage, possibly forming neutron stars or black holes.

In a neutron star, we can use the white dwarf calculations to work out the star's radius, since neutrons are fermions. The only differences in the calculation are: (1) use the neutron mass m_n in place of the electron mass and (2) take $q = 1$ instead of $\frac{1}{2}$ since there is only one fermion (the neutron itself) per nucleon. The original formula for the radius of a white dwarf is

$$R = \left(\frac{9\pi}{4}\right)^{2/3} \frac{\hbar^2 q^{5/3}}{Gmm_n^2 N^{1/3}} \tag{0.14}$$

so we can take this result and multiply it by $2^{5/3} m/m_n$. The radius of a solar-mass white dwarf is 7162 km, so the radius of a solar-mass neutron star is

$$R = 7162 \frac{2^{5/3} m}{m_n} = 12.38 \text{ km} \tag{0.15}$$

For the Fermi energy, we make the same two replacements

$$E_F = \frac{\hbar^2 k_F^2}{2m_n} \tag{0.16}$$

$$= \frac{\hbar^2}{2m_n}\left(\frac{3\pi^2 Nq}{V}\right)^{2/3} \tag{0.17}$$

$$= \frac{\hbar^2}{2m_n}\left(\frac{9\pi N}{4R^3}\right)^{2/3} \tag{0.18}$$

$$= 55.7 \text{ MeV} \tag{0.19}$$

where we used $N = 1.189 \times 10^{57}$ for the sun (as we found in the white dwarf calculation). The rest mass of a nucleon is about 940 MeV, so we're not really in the relativistic zone.

Incidentally, the density of a neutron star is about $2.5 \times 10^{17}\text{kg m}^{-3}$ which is more than 10^{14} times the density of water.