BOSONS IN THE INFINITE SQUARE WELL

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 6.3.

An example of perturbation theory applied to the interaction between two particles. We place two bosons in an infinite square well, where the single particle wave functions are given by

$$\psi_n = \sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a} \tag{1}$$

and energies by

$$E_n = \frac{(n\pi\hbar)^2}{2ma^2} \tag{2}$$

Since there is no exclusion principle for bosons, both particles can be in the ground state, with wave function

$$\psi_{11} = \frac{2}{a}\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a} \tag{3}$$

The energy of this state is

$$E_{11,0} = 2E_{1,0} = \frac{\pi^2\hbar^2}{ma^2} \tag{4}$$

If one boson is in the ground state and the other in the first excited state, the overall wave function must be symmetric with respect to interchange of the particles, so we have

$$\psi_{12} = \frac{\sqrt{2}}{a}\left[\sin\frac{2\pi x_1}{a}\sin\frac{\pi x_2}{a} + \sin\frac{\pi x_1}{a}\sin\frac{2\pi x_2}{a}\right] \tag{5}$$

Applying the two-particle hamiltonian to this wave function yields the total energy as the sum of the two individual energies:

$$E_{12,0} = E_{1,0} + E_{2,0} = \frac{5}{2}\frac{\pi^2\hbar^2}{ma^2} \tag{6}$$

Now suppose we introduce an interaction between the particles of form

$$V = -aV_0\delta(x_1 - x_2) \tag{7}$$

That is, there is a delta function well when the particles occupy the same location. Using first order perturbation theory, the effect on the energy $E_{11,0}$ is

$$E_{11,1} = \langle 11,0|\,V\,|11,0\rangle \tag{8}$$

$$= -aV_0\frac{4}{a^2}\int_0^a\int_0^a \sin^2\frac{\pi x_1}{a}\sin^2\frac{\pi x_2}{a}\delta\left(x_1-x_2\right)dx_1dx_2 \tag{9}$$

$$= -\frac{4V_0}{a}\int_0^a \sin^4\frac{\pi x_1}{a}dx_1 \tag{10}$$

$$= -\frac{3}{2}V_0 \tag{11}$$

For $E_{12,0}$ we have

$$E_{12,1} = \langle 12,0|\,V\,|12,0\rangle \tag{12}$$

$$= -aV_0\frac{2}{a^2}\int_0^a 4\sin^2\frac{\pi x_1}{a}\sin^2\frac{2\pi x_1}{a}dx_1 \tag{13}$$

$$= -2V_0 \tag{14}$$

We used Maple for the integral.