HARMONIC OSCILLATOR IN AN ELECTRIC FIELD

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 6.5.

As another example of second order perturbation theory we start with the harmonic oscillator potential, except this time we place a charged particle (charge q) into the potential and turn on a small electric field E, so that the perturbation in the potential is

$$V=-qEx \tag{1}$$

We'll begin by looking at the first order correction, for which we have

$$E_{n1}=\langle n0|V|n0\rangle \tag{2}$$

From our earlier calculations of the matrix elements in the harmonic oscillator potential, we have

$$\langle n0|x|j0\rangle=\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{j+1}\delta_{n,j+1}+\sqrt{j}\delta_{n,j-1}\right) \tag{3}$$

Since all diagonal elements are zero, $\langle n0|V|n0\rangle=0$ and there is no first order correction to the energy.

There are only two non-zero terms in the second order correction series

$$E_{n2}=\sum_{j\neq n}\frac{|\langle j0|V|n0\rangle|^2}{E_{n0}-E_{j0}} \tag{4}$$

That is, the terms for $j=n\pm 1$. Since $E_{n0}=\hbar\omega\left(n+\frac{1}{2}\right)$, we get

$$\begin{aligned}
E_{n2} &= \frac{\hbar q^2E^2}{2m\omega}\left(\frac{-(n+1)+n}{\hbar\omega}\right) && (5)\\
&= -\frac{q^2E^2}{2m\omega^2} && (6)\\
&= -\frac{q^2E^2}{2k} && (7)
\end{aligned}$$

In this case, we can actually find an exact solution for the perturbed potential. The Schrödinger equation for the perturbed potential is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\left(\frac{1}{2}kx^2-qEx\right)\psi=E_n\psi \tag{8}$$

Multiplying through by $2/k$ we get

$$-\frac{2}{k}\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+x\left(x-\frac{qE}{k}-\frac{qE}{k}\right)\psi=\frac{2}{k}E_n\psi \tag{9}$$

We can now make the variable transformation

$$\xi\equiv x-\frac{qE}{k} \tag{10}$$

Since $d\xi=dx$, this transforms the Schrödinger equation to

$$-\frac{2}{k}\frac{\hbar^2}{2m}\frac{d^2\psi}{d\xi^2}+\left(\xi+\frac{qE}{k}\right)\left(\xi-\frac{qE}{k}\right)\psi=\frac{2}{k}E_n\psi \tag{11}$$

$$-\frac{2}{k}\frac{\hbar^2}{2m}\frac{d^2\psi}{d\xi^2}+\left(\xi^2-\left(\frac{qE}{k}\right)^2\right)\psi=\frac{2}{k}E_n\psi \tag{12}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{d\xi^2}+\frac{1}{2}k\xi^2\psi=\left(E_n+\frac{q^2E^2}{2k}\right)\psi \tag{13}$$

The last line has the same form as the Schrödinger equation for the unperturbed potential except that the energies are given by $E_n+\frac{q^2E^2}{2k}$, which is a constant (independent of ξ), so the solutions must be the same as for the unperturbed oscillator. That is

$$E_n+\frac{q^2E^2}{2k}=\hbar\omega\left(n+\frac{1}{2}\right) \tag{14}$$

$$E_n=\hbar\omega\left(n+\frac{1}{2}\right)-\frac{q^2E^2}{2k} \tag{15}$$

The correction to the energy is the same as the second order term found above, so in this case, second order perturbation theory gives the *exact* correction to the energy.