

HARMONIC OSCILLATOR IN AN ELECTRIC FIELD

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 6.5.

As another example of second order perturbation theory we start with the harmonic oscillator potential, except this time we place a charged particle (charge q) into the potential and turn on a small electric field E , so that the perturbation in the potential is

$$(0.1) \quad V = -qEx$$

We'll begin by looking at the first order correction, for which we have

$$(0.2) \quad E_{n1} = \langle n0 | V | n0 \rangle$$

From our earlier calculations of the matrix elements in the harmonic oscillator potential, we have

$$(0.3) \quad \langle n0 | x | j0 \rangle = \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{j+1} \delta_{n,j+1} + \sqrt{j} \delta_{n,j-1} \right)$$

Since all diagonal elements are zero, $\langle n0 | V | n0 \rangle = 0$ and there is no first order correction to the energy.

There are only two non-zero terms in the second order correction series

$$(0.4) \quad E_{n2} = \sum_{j \neq n} \frac{|\langle j0 | V | n0 \rangle|^2}{E_{n0} - E_{j0}}$$

That is, the terms for $j = n \pm 1$. Since $E_{n0} = \hbar\omega \left(n + \frac{1}{2} \right)$, we get

$$(0.5) \quad E_{n2} = \frac{\hbar q^2 E^2}{2m\omega} \left(\frac{-(n+1) + n}{\hbar\omega} \right)$$

$$(0.6) \quad = -\frac{q^2 E^2}{2m\omega^2}$$

$$(0.7) \quad = -\frac{q^2 E^2}{2k}$$

In this case, we can actually find an exact solution for the perturbed potential. The Schrödinger equation for the perturbed potential is

$$(0.8) \quad -\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + \left(\frac{1}{2} k x^2 - q E x \right) \psi = E_n \psi$$

Multiplying through by $2/k$ we get

$$(0.9) \quad -\frac{2}{k} \frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + x \left(x - \frac{qE}{k} - \frac{qE}{k} \right) \psi = \frac{2}{k} E_n \psi$$

We can now make the variable transformation

$$(0.10) \quad \xi \equiv x - \frac{qE}{k}$$

Since $d\xi = dx$, this transforms the Schrödinger equation to

$$(0.11) \quad -\frac{2}{k} \frac{\hbar^2}{2m} \frac{d^2 \psi}{d\xi^2} + \left(\xi + \frac{qE}{k} \right) \left(\xi - \frac{qE}{k} \right) \psi = \frac{2}{k} E_n \psi$$

$$(0.12) \quad -\frac{2}{k} \frac{\hbar^2}{2m} \frac{d^2 \psi}{d\xi^2} + \left(\xi^2 - \left(\frac{qE}{k} \right)^2 \right) \psi = \frac{2}{k} E_n \psi$$

$$(0.13) \quad -\frac{\hbar^2}{2m} \frac{d^2 \psi}{d\xi^2} + \frac{1}{2} k \xi^2 \psi = \left(E_n + \frac{q^2 E^2}{2k} \right) \psi$$

The last line has the same form as the Schrödinger equation for the unperturbed potential except that the energies are given by $E_n + \frac{q^2 E^2}{2k}$, which is a constant (independent of ξ), so the solutions must be the same as for the unperturbed oscillator. That is

$$(0.14) \quad E_n + \frac{q^2 E^2}{2k} = \hbar \omega \left(n + \frac{1}{2} \right)$$

$$(0.15) \quad E_n = \hbar \omega \left(n + \frac{1}{2} \right) - \frac{q^2 E^2}{2k}$$

The correction to the energy is the same as the second order term found above, so in this case, second order perturbation theory gives the *exact* correction to the energy.