PERTURBING A PARTICLE ON A CIRCULAR WIRE

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 6.7.

Here's another example of applying degenerate pertrubation theory. Suppose we revisit the problem of a particle with a periodic wave function: the particle of mass m that slides on a frictionless circular wire of circumference L. We saw that the stationary states are

$$|n0\rangle = \frac{1}{\sqrt{L}} e^{2n\pi xi/L} \tag{1}$$

and the energies are

$$E_{\pm n0} = \frac{2n^2\pi^2\hbar^2}{mL^2} \tag{2}$$

The system is doubly degenerate for all values of n except 0, so the degenerate perturbation theory applies.

Now we introduce the perturbation

$$V = -V_0 e^{-x^2/a^2} \tag{3}$$

where V_0 is assumed to be small compared with the unperturbed energies, and $a \ll L$. Since the unperturbed hamiltonian is that of the free particle (with the periodic constraint), the perturbation is the only non-zero part of the potential.

We can find the first-order perturbations to the energy using the formula we derived in the last post, with the suffixes on the matrix elements customized to the two degenerate states for $E_{\pm n0}$:

$$E_{n1} = \frac{1}{2}\left(W_{nn} + W_{-n,-n} \pm \sqrt{(W_{nn} - W_{-n,-n})^2 + 4\left|W_{n,-n}\right|^2}\right) \tag{4}$$

The matrix elements can be calculated as:

$$W_{qr} = \langle q0|V|r0\rangle \tag{5}$$

$$= -\frac{V_0}{L}\int_{-L/2}^{L/2} e^{2(r-q)\pi xi/L} e^{-x^2/a^2}dx \tag{6}$$

Because of the assumption $a \ll L$, the e^{-x^2/a^2} term becomes very small for relatively small values of x, so we can approximate the integral by extending the limits to infinity in both directions. Using software, this integral comes out to

$$W_{qr} = -\frac{\sqrt{\pi}aV_0}{L}e^{-a^2\pi^2(r-q)^2/L^2} \tag{7}$$

The required matrix elements are then

$$W_{nn} = W_{-n,-n} = -\frac{\sqrt{\pi}aV_0}{L} \tag{8}$$

$$W_{n,-n} = -\frac{\sqrt{\pi}aV_0}{L}e^{-4a^2\pi^2n^2/L^2} \tag{9}$$

The two perturbations on the unperturbed energy $E_{\pm n0}$ are

$$E_{n1} = -\frac{\sqrt{\pi}aV_0}{L}\left(1 \pm e^{-4a^2\pi^2n^2/L^2}\right) \tag{10}$$

Since $W_{n,-n} \neq 0$, the states ψ_{n0} are not the 'special' states we mentioned in the last post. We can find the special states as follows. We know that these special states are linear combinations of ψ_{+n0} and ψ_{-n0}, so let's define the special states as

$$|a0\rangle = \alpha_a|+n0\rangle + \beta_a|-n0\rangle \tag{11}$$

$$|b0\rangle = \alpha_b|+n0\rangle + \beta_b|-n0\rangle \tag{12}$$

Since $W_{ab} = 0$ for these states, we have

$$W_{ab} = \langle a0|V|b0\rangle \tag{13}$$

$$= \alpha_a^*\alpha_b W_{nn} + \alpha_a^*\beta_b W_{n,-n} + \beta_a^*\alpha_b W_{-n,n} + \beta_a^*\beta_b W_{-n,-n} \tag{14}$$

$$= 0 \tag{15}$$

Substituting the values for the Ws from above, we get

$$\alpha_a^*\alpha_b + \beta_a^*\beta_b + e^{-4a^2\pi^2n^2/L^2}\left(\alpha_a^*\beta_b + \beta_a^*\alpha_b\right) = 0 \tag{16}$$

We can satisfy this equation for all n if we choose

$$\alpha_a = -\beta_a \tag{17}$$
$$\alpha_b = \beta_b \tag{18}$$

With the normalization condition $|\alpha_i|^2 + |\beta_i|^2 = 1$ we get the final form of the special states:

$$|a0\rangle = \frac{1}{\sqrt{2}}\left(|{+n0}\rangle - |{-n0}\rangle\right) \tag{19}$$
$$|b0\rangle = \frac{1}{\sqrt{2}}\left(|{+n0}\rangle + |{-n0}\rangle\right) \tag{20}$$

For the special states, we can get the energy perturbations using the formula from nondegenerate theory

$$E_{n1,a} = W_{aa} \tag{21}$$
$$= \frac{1}{2}\left[W_{nn} + W_{-n,-n} - 2W_{n,-n}\right] \tag{22}$$
$$= -\frac{\sqrt{\pi}aV_0}{L}\left(1 - e^{-4a^2\pi^2n^2/L^2}\right) \tag{23}$$
$$E_{n1,b} = W_{bb} \tag{24}$$
$$= \frac{1}{2}\left[W_{nn} + W_{-n,-n} + 2W_{n,-n}\right] \tag{25}$$
$$= -\frac{\sqrt{\pi}aV_0}{L}\left(1 + e^{-4a^2\pi^2n^2/L^2}\right) \tag{26}$$

These are the same energies as before.

Finally, it's worth quoting a theorem that makes finding these special states a little easier in some cases. If you can find some hermitian operator A that commutes with *both* the unperturbed hamiltonian H_0 and the perturbation V and you can find two eigenfunctions that have the same eigenvalue under H_0 but *different* eigenvalues under A, then these eigenfunctions are the special states. That is, we look for functions $|a0\rangle$ and $|b0\rangle$ Such that

$$H_0|a0\rangle = E|a0\rangle \tag{27}$$
$$H_0|b0\rangle = E|b0\rangle \tag{28}$$
$$A|a0\rangle = A_a|a0\rangle \tag{29}$$
$$A|b0\rangle = A_b|b0\rangle \tag{30}$$

where $A_a \neq A_b$. To see this, we use the assumption that $[V,A] = 0$ so that

$$\langle a0|[V,A]|b0\rangle = \langle a0|VA|b0\rangle - \langle a0|AV|b0\rangle \tag{31}$$
$$= (A_b - A_a)\langle a0|V|b0\rangle \tag{32}$$
$$= (A_b - A_a)W_{ab} \tag{33}$$
$$= 0 \tag{34}$$

Since we're assuming $A_a \neq A_b$, we must have $W_{ab} = 0$.

Actually finding such an operator A can be a bit tricky however. In the current example, we can't use an operator that has a derivative with respect to x, since that wouldn't commute with V. Likewise, we can't use an operator that involves just multiplying by a function of x, since that wouldn't commute with H_0 (which has a derivative in it). After some searching, it seems that we can use the parity operator which has the effect of reversing the sign of x: $Af(x) = f(-x)$. It's not at all clear to me how you can arrive at this answer through any process of logical deduction, however.

To see that it works, we can look at the special functions we derived above:

$$|a0\rangle = \frac{1}{\sqrt{2}}(|+n0\rangle - |-n0\rangle) = \sqrt{\frac{2}{L}}i\sin\frac{2\pi nx}{L} \tag{35}$$
$$|b0\rangle = \frac{1}{\sqrt{2}}(|+n0\rangle + |-n0\rangle) = \sqrt{\frac{2}{L}}\cos\frac{2\pi nx}{L} \tag{36}$$

Under the parity operator, we have

$$A|a0\rangle = -|a0\rangle \tag{37}$$
$$A|b0\rangle = |b0\rangle \tag{38}$$

Thus the eigenvalues of A are ± 1 and the conditions are satisfied. However, it's not exactly clear that if we *didn't* know $|a0\rangle$ and $|b0\rangle$ beforehand that we would hit on the idea of using the parity operator, and then manage to find its eigenfunctions. The straightforward route taken above (using the quadratic formula to find the energies) is at least always reliable, even if it might take a bit more calculation.

PINGBACKS

Pingback: Perturbation theory for higher-level degenerate systems
Pingback: Fine structure of hydrogen: relativistic correction
Pingback: Fine structure of hydrogen: spin-orbit coupling

Pingback: Hyperfine splitting and the 21 cm line of hydrogen