HARMONIC OSCILLATOR: RELATIVISTIC CORRECTION

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 6.14.

Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 7.4, Exercise 7.4.4.

We can apply the relativistic correction to the one-dimensional harmonic oscillator as another example. When analyzing the hydrogen atom, we arrived at this formula for the first order correction to the energy:

$$E_{n1} = -\frac{1}{2mc^2}\langle n0|\left(E_{n0}-V\right)^2|n0\rangle \tag{1}$$

where we've adjusted the wave functions so they apply to the harmonic oscillator.

Before applying this formula, we should check a couple of things. First, this formula was derived using non-degenerate perturbation theory. In the one-dimensional oscillator this is fine, since there are no degenerate states.

Second, we assumed that the operator p^4 was hermitian, and to check this it is easiest to use the raising and lowering operators. We have

$$p = i\sqrt{\frac{\hbar m\omega}{2}}\left(a_+ - a_-\right) \tag{2}$$

The raising and lowering operators transform one wave function into another:

$$a_+|n0\rangle = \sqrt{n+1}\,|n+1,0\rangle \tag{3}$$

$$a_-|n0\rangle = \sqrt{n}\,|n-1,0\rangle \tag{4}$$

Therefore, each application of p transforms the original wave function into a linear combination of other wave functions and since p itself must be hermitian (it represents an observable: the momentum) when applied to any oscillator wave function, any power of p is also hermitian in the same situation.

Having verified that the first order energy correction may be applied to the harmonic oscillator, we can now plug in the values. The unperturbed energies are

$$E_{n0}=\left(n+\frac{1}{2}\right)\hbar\omega \tag{5}$$

From the virial theorem we know that $\langle T\rangle=\langle V\rangle=\frac{1}{2}E_{n0}$ so

$$E_{n1}=-\frac{1}{2mc^2}\left(E_{n0}^2-2E_{n0}\langle V\rangle+\left\langle V^2\right\rangle\right) \tag{6}$$

$$=-\frac{1}{2mc^2}\left[\left(\left(n+\frac{1}{2}\right)\hbar\omega\right)^2-\left(\left(n+\frac{1}{2}\right)\hbar\omega\right)^2+\frac{1}{4}m^2\omega^4\left\langle x^4\right\rangle\right] \tag{7}$$

$$=-\frac{m\omega^4}{8c^2}\left\langle x^4\right\rangle \tag{8}$$

To calculate $\left\langle x^4\right\rangle$, we can use the raising and lowering operators again. We have

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \tag{9}$$

$$x^2 = \frac{\hbar}{2m\omega}\left(a_+^2+a_+a_-+a_-a_++a_-^2\right) \tag{10}$$

Since $\left\langle x^4\right\rangle=\langle n0|x^4|n0\rangle$, the two wave functions involved in calculating the mean value are the same (both $|n0\rangle$) and $\langle n0|\,m0\rangle=\delta_{mn}$, any combination of a_+ and a_- that converts $|n0\rangle$ into a different wave function will not contribute to the overall integral, so we need consider only those terms in the operator x^4 with equal numbers of a_+ and a_-. Retaining only these terms, we get

$$x^4=\left(\frac{\hbar}{2m\omega}\right)^2\left(a_+^2a_-^2+a_+a_-a_+a_-+a_+a_-^2a_++a_-a_+a_-a_++a_-a_+^2a_-+a_-^2a_+^2\right) \tag{11}$$

Applying the operators according to the formulas above, we get

$$\left\langle x^4\right\rangle=\left(\frac{\hbar}{2m\omega}\right)^2\left[n(n-1)+n^2+n(n+1)+(n+1)^2+n(n+1)+(n+1)(n+2)\right] \tag{12}$$

$$=\left(\frac{\hbar}{2m\omega}\right)^2\left(6n^2+6n+3\right) \tag{13}$$

The energy correction is then

$$E_{n1} = -\frac{3\hbar^2\omega^2}{32mc^2}\left(2n^2 + 2n + 1\right) \tag{14}$$