FINE STRUCTURE OF HYDROGEN: DIRAC FORMULA

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The fine structure of hydrogen as worked out from applying relativistic and spin-orbit coupling corrections using perturbation theory gives a formula for the energy levels as follows:

\[
E_{nj} = E_{n0} \left[ 1 - \frac{\alpha^2}{4n^2} \left( 3 - \frac{4n}{j + \frac{1}{2}} \right) \right] 
\]

\[
= -\frac{13.6 \text{ eV}}{n^2} \left[ 1 - \frac{\alpha^2}{4n^2} \left( 3 - \frac{4n}{j + \frac{1}{2}} \right) \right] 
\]

If the hydrogen atom is analyzed using the relativistic Dirac equation (which we’ll hopefully get to one day :) ), we can get an exact formula for the energy levels:

\[
E_{nj} = mc^2 \left[ 1 + \left( \frac{\alpha}{n - j - \frac{1}{2} + \sqrt{(j + \frac{1}{2})^2 - \alpha^2}} \right) \right]^{-1/2} - 1 \]

(3)

Here \( n \) is the principal quantum number, \( j \) is the total angular momentum quantum number and \( \alpha \) is the fine structure constant. We can expand this in a Taylor series in \( \alpha \). The derivatives are tedious and can be done using Maple, with the result

\[
E_{nj} = \frac{-1}{2} \frac{mc^2}{n^2} \alpha^2 - \frac{1}{8} \frac{mc^2}{(1+2j)n^4} (8n-6j-3) \alpha^4 + \mathcal{O}(\alpha^6) 
\]

(4)

We can write the unperturbed energy \( E_{n0} \) in terms of \( \alpha \):

\[
E_n = -\frac{m}{2n^2} \left( \frac{e^2}{4\pi\epsilon_0\hbar} \right)^2 = -\frac{1}{2n^2} \alpha^2 mc^2 
\]

(5)

Using this, we can write out the expansion as
Thus the first two non-zero terms in the Taylor expansion of the Dirac formula give us the formula using perturbation theory in the non-relativistic analysis.