ZEEMAN EFFECT: WEAK FIELD

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 6.21.

We saw that the Zeeman effect can be divided into 3 cases: weak, strong or intermediate external field. We now need to consider how to calculate the effects of the external field on the energy levels.

There are effectively two magnetic dipoles in the hydrogen atom (well, 3 if you count the proton's spin, but we'll get to that later; here we're concerned only with effects on the electron): that from the orbital angular momentum and that from spin. The energy associated with a dipole μ in an external magnetic field is

$$U = -\mu \cdot \mathbf{B}_{ext} \tag{1}$$

so the Zeeman hamiltonian component is

$$H_Z' = -(\mu_L + \mu_S) \cdot \mathbf{B}_{ext} \tag{2}$$

To work out these magnetic moments (well, the spin one, anyway) properly requires relativistic quantum theory so we'll just quote the results:

$$\mu_L = -\frac{e}{2m}\mathbf{L} \tag{3}$$

$$\mu_S = -\frac{e}{m}\mathbf{S} \tag{4}$$

If we direct $\mathbf{B}_{ext}$ along the z axis, then

$$H_Z' = \frac{e}{2m}(L_z + 2S_z) B_{ext} \tag{5}$$

It's fairly obvious from this that H_Z' commutes with L_z and S_z and, since L^2 and S^2 both commute with L_z and S_z as well, H_Z' commutes with these operators also. However, if we look at the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$, then the operator $J^2 = L^2 + S^2 + 2\mathbf{S} \cdot \mathbf{L}$ does *not* commute with L_z and S_z because of the $\mathbf{S} \cdot \mathbf{L}$ term. For example,

$$[S_xL_x, L_z] = S_x [L_x, L_z] = -i\hbar S_xL_y \quad (6)$$

$$[S_xL_x, S_z] = L_x [S_x, S_z] = -i\hbar L_xS_y \quad (7)$$

and so on. This result may seem bizarre: how can $\mathbf{L}$ and $\mathbf{S}$ independently be conserved, yet their sum $\mathbf{J}$ not be conserved?

To see how this works, we consider the effect of $\mathbf{B}_{ext}$ on the two dipole moments. The external field exerts a torque on each moment, which is given by $\mathbf{N} = \mu \times \mathbf{B}_{ext}$. Suppose μ makes an angle θ with $\mathbf{B}_{ext}$. Then the magnitude of the torque is $N = \mu B_{ext} \sin\theta$ and its direction is perpendicular to both the moment and the field. This causes μ to precess around $\mathbf{B}_{ext}$ with θ remaining constant and the azimuthal angle ϕ (in spherical coordinates) changing at a constant rate. This is true for both μ_L and μ_S, but since the magnitudes of these moments (and the angles θ) will, in general, be different for the two moments, the rate of precession will vary causing the moments (and thus the angular momenta $\mathbf{L}$ and $\mathbf{S}$ associated with them) to point in different relative directions as time goes by. Thus the vector sum $\mathbf{J} = \mathbf{L} + \mathbf{S}$ will vary, even though the magnitudes of $\mathbf{L}$ and $\mathbf{S}$ and their z components L_z and S_z remain constant.

This discussion ignores spin-orbit coupling, however. If we remove $\mathbf{B}_{ext}$ and instead consider just the behaviour of the electron's spin magnetic moment in the field $\mathbf{B}_L$ produced by the proton's relative orbit around it, then $\mathbf{B}_p$ exerts a torque on μ_S causing it to precess about $\mathbf{B}_L$. However, the spin also produces a magnetic field $\mathbf{B}_S$ which exerts an equal and opposite torque on the orbital moment, with the result that μ_L precesses about $\mathbf{B}_S$. In our analysis of spin-orbit coupling, we saw that J^2 and J_z commute with H'_{so} while L_z and S_z do not. What is happening here is that the sum $\mathbf{J} = \mathbf{L} + \mathbf{S}$ is constant, but $\mathbf{L}$ and $\mathbf{S}$ each precess about $\mathbf{J}$ in such a way as to conserve J^2 and J_z, but with varying L_z and S_z.

I have to confess that I haven't found a rigorous derivation of these results, so it's possible that they are merely classical analogies designed to make the quantum mechanics easier to understand. Since the components of angular momentum don't all have well-defined values at any given time (since they don't commute), the notion of a rigid vector precessing around an axis isn't valid anyway. If anyone knows of a better derivation or explanation, please do let me know.

The question when analyzing the Zeeman effect is which of these two processes (the external field or spin-orbit coupling) is dominant, since this affects how we do the analysis. In the weak field case, spin-orbit coupling produces a vector $\mathbf{J}$ which will then precess about $\mathbf{B}_{ext}$, while in the strong field case, the field essentially breaks the coupling between **L** and **S** and forces the two moments to act independently. We'll have a look at the weak field case here.

In this case, spin-orbit coupling is the dominant perturbation, with the Zeeman effect being a smaller perturbation on top of the spin-orbit perturbation. Since the spin-orbit effect is dominant, we use the eigenstates of that hamiltonian to calculate the first-order corrections due to the Zeeman effect. This isn't strictly valid since as we saw above, J^2 and J_z are 'good' operators for spin-orbit coupling but not for the Zeeman effect, so technically we should use degenerate perturbation theory to calculate the latter. However, a decent approximation is to use the spin-orbit eigenstates to calculate the Zeeman corrections.

Since L_z and S_z are not 'good' operators for spin-orbit coupling, we go back to the original Zeeman hamiltonian 2 in terms of angular momenta:

$$H_Z' = \frac{e}{2m}(\mathbf{L}+2\mathbf{S})\cdot\mathbf{B}_{ext} = \frac{e}{2m}(\mathbf{J}+\mathbf{S})\cdot\mathbf{B}_{ext} \tag{8}$$

The eigenstates we'll use are $|n\ell j j_z\rangle$ so the Zeeman energy correction is

$$\begin{aligned}
E_{Z1} &= \langle n\ell j j_z| H_Z' |n\ell j j_z\rangle & (9)\\
&= \frac{eB_{ext}}{2m}\langle n\ell j j_z| J_z |n\ell j j_z\rangle + \frac{e}{2m}\langle n\ell j j_z| \mathbf{S}\cdot\mathbf{B}_{ext} |n\ell j j_z\rangle & (10)\\
&= \frac{eB_{ext}}{2m}\hbar j_z + \frac{e}{2m}\langle \mathbf{S}\cdot\mathbf{B}_{ext}\rangle & (11)\\
&= \frac{eB_{ext}}{2m}\hbar j_z + \frac{e}{2m}\langle \mathbf{S}\rangle\cdot\mathbf{B}_{ext} & (12)
\end{aligned}$$

The first term in the second line assumes that $\mathbf{B}_{ext}$ points along the z axis. The last term involves the average projection of **S** onto the z axis which is, of course, S_z but this doesn't do us much good here since the states are not eigenstates of S_z. The argument at this point makes use of the fact that **S** precesses about the fixed total angular momentum **J** and therefore its time average is the projection of **S** onto **J**, since the component of **S** perpendicular to **J** averages out to zero. That is

$$\langle\mathbf{S}\rangle = \left\langle \frac{\mathbf{S}\cdot\mathbf{J}}{J^2}\mathbf{J}\right\rangle \tag{13}$$

so

$$E_{Z1} = \frac{eB_{ext}}{2m}\hbar j_z + \frac{e}{2m}\left\langle\frac{\mathbf{S}\cdot\mathbf{J}}{J^2}\right\rangle\langle\mathbf{J}\cdot\mathbf{B}_{ext}\rangle \tag{14}$$

$$= \frac{eB_{ext}}{2m}\hbar j_z\left(1+\left\langle\frac{\mathbf{S}\cdot\mathbf{J}}{J^2}\right\rangle\right) \tag{15}$$

To get the final average, we use

$$\mathbf{L} = \mathbf{J}-\mathbf{S} \tag{16}$$

$$L^2 = J^2+S^2-2\mathbf{S}\cdot\mathbf{J} \tag{17}$$

$$\mathbf{S}\cdot\mathbf{J} = \frac{1}{2}\left(J^2+S^2-L^2\right) \tag{18}$$

or in terms of the eigenvalues

$$\langle\mathbf{S}\cdot\mathbf{J}\rangle = \frac{\hbar^2}{2}\left(j(j+1)+\frac{3}{4}-\ell(\ell+1)\right) \tag{19}$$

Putting it all together, we get

$$E_{Z1} = \frac{e\hbar B_{ext}}{2m} j_z\left(1+\frac{j(j+1)+\frac{3}{4}-\ell(\ell+1)}{2j(j+1)}\right) \tag{20}$$

This expression is often condensed to

$$E_{Z1} = \mu_B g_J B_{ext} j_z \tag{21}$$

$$\mu_B \equiv \frac{e\hbar}{2m} \tag{22}$$

$$g_J \equiv 1+\frac{j(j+1)+\frac{3}{4}-\ell(\ell+1)}{2j(j+1)} \tag{23}$$

where μ_B is called the Bohr magneton and g_J is called the Landé g-factor. Combining the Zeeman formula with the fine structure formula, we get the overall energy states of hydrogen:

$$E_n = -\frac{13.6\text{ eV}}{n^2}\left[1-\frac{\alpha^2}{4n^2}\left(3-\frac{4n}{j+\frac{1}{2}}\right)\right]+\mu_B g_J B_{ext} j_z \tag{24}$$

For given values of ℓ and j, the Zeeman effect splits an energy level into separate levels, one for each possible value of j_z. Plotting the energy E_n versus $\mu_B B_{ext}$ we get one straight line with slope $g_J j_z$ for each possible value of j_z. For example, for $n=2$ the possibilities are

ℓ	j	j_z	g_J	Energy (eV)	Slopes
0	$\frac{1}{2}$	$\pm\frac{1}{2}$	2	$-3.4-1.0625\alpha^2\pm\mu_B B_{ext}$	± 1
1	$\frac{1}{2}$	$\pm\frac{1}{2}$	$\frac{2}{3}$	$-3.4-1.0625\alpha^2\pm\frac{1}{3}\mu_B B_{ext}$	$\pm\frac{1}{3}$
1	$\frac{3}{2}$	$\pm\frac{1}{2}$	$\frac{4}{3}$	$-3.4-0.2125\alpha^2\pm\frac{2}{3}\mu_B B_{ext}$	$\pm\frac{2}{3}$
1	$\frac{3}{2}$	$\pm\frac{3}{2}$	$\frac{4}{3}$	$-3.4-0.2125\alpha^2\pm 2\mu_B B_{ext}$	± 2

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