

PERTURBATION DUE TO FINITE SIZE OF THE PROTON IN HYDROGEN

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 6.29.

All the corrections to the spectrum of hydrogen that we've considered so far assume that the nucleus is a point charge, so the Coulomb potential applies at all distances from the proton. In reality, the proton does have a finite, non-zero radius, so here we do a crude calculation to see what effect this might have on the energy levels.

The model takes the proton to be a spherical shell of charge of radius b . By Gauss's law and the spherical symmetry of the model, the electric field inside the shell is zero, so the electric potential inside is constant and equal to its value at the surface of the sphere, namely $e/4\pi\epsilon_0 b$, meaning that the potential energy of the electron inside the sphere is

$$(1) \quad E = -\frac{e^2}{4\pi\epsilon_0 b}$$

Since the point-nucleus model assumes the potential energy is $-\frac{e^2}{4\pi\epsilon_0 r}$ all the way down to $r = 0$, the correction to the hamiltonian is

$$(2) \quad H' = \begin{cases} \frac{e^2}{4\pi\epsilon_0} \left(\frac{1}{r} - \frac{1}{b}\right) & r \leq b \\ 0 & r \geq b \end{cases}$$

Looking at the non-degenerate ground state of hydrogen, the unperturbed wave function is

$$(3) \quad \psi_{100} = \frac{2}{\sqrt{4\pi a^3}} e^{-r/a}$$

so the first order perturbation in the energy is (the angular integrals contribute a factor of 4π):

$$(4) \quad \Delta E = \frac{e^2}{\pi a^3 \epsilon_0} \int_0^b \left(\frac{1}{r} - \frac{1}{b} \right) e^{-2r/a} r^2 dr$$

$$(5) \quad = \frac{e^2}{4\pi a \epsilon_0} \left(\frac{a}{b} \left(e^{-2b/a} - 1 \right) + e^{-2b/a} + 1 \right)$$

$$(6) \quad = \frac{e^2}{4\pi a \epsilon_0} \left[\frac{2}{3} \left(\frac{b}{a} \right)^2 + \mathcal{O} \left(\frac{b^3}{a^3} \right) \right]$$

where in the last line, we expanded the exponentials.

The unperturbed ground state energy is

$$(7) \quad E_{1,0} = -\frac{m}{2\hbar^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 = -\frac{e^2}{8\pi\epsilon_0 a}$$

To first order

$$(8) \quad \Delta E = -\frac{4}{3} E_{1,0} \left(\frac{b}{a} \right)^2$$

and the relative shift in energy due to the finite size of the proton is

$$(9) \quad \frac{\Delta E}{E_{1,0}} = -\frac{4}{3} \left(\frac{b}{a} \right)^2$$

Plugging in the numbers

$$(10) \quad a = 5.29 \times 10^{-11} \text{ m}$$

$$(11) \quad b = 10^{-15} \text{ m}$$

$$(12) \quad \frac{\Delta E}{E_{1,0}} = -4.76 \times 10^{-10}$$

For comparison, the fine structure shift is (for $n = 1$ and $j = \frac{1}{2}$):

$$(13) \quad \frac{\Delta E_{fs}}{E_{1,0}} = \frac{\alpha^2}{4} = 1.33 \times 10^{-5}$$

The hyperfine shift is

$$(14) \quad \frac{\Delta E_{hs}}{E_{1,0}} = \frac{5.88 \times 10^{-6}}{13.6} = 4.32 \times 10^{-7}$$

Thus the shift due to the finite size of the proton is much smaller than even the hyperfine shift.