VARIATIONAL PRINCIPLE AND THE ELECTRON IN A MAGNETIC FIELD

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 7.16.

As an example of the two-state system we looked at earlier, we can consider an electron in a magnetic field. The hamiltonian for a spin-1/2 particle such as the electron in a magnetic field is

$$\mathsf{H} = -\gamma \mathbf{B} \cdot \mathsf{S} \tag{1}$$

where S is the spin matrix and $\gamma = -e/m$ is the gyromagnetic ratio of the electron.

We start with a uniform magnetic field in the z direction $\mathbf{B} = B_z \hat{\mathbf{z}}$, for which the hamiltonian is

$$\mathsf{H} = \frac{e}{m} B_z S_z = \frac{eB_z \hbar}{2m} \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right] \tag{2}$$

The energies are

$$E_{b,a} = \pm \frac{eB_z \hbar}{2m} \tag{3}$$

(taking $E_b > E_a$ as before).

We now introduce a perturbation by turning on a weak field in the x direction $\mathbf{B}' = B_x \hat{\mathbf{x}}$. The perturbation in the hamiltonian is

$$H' = \frac{eB_x \hbar}{2m} \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \tag{4}$$

This has the same form as the perturbation in the previous problem:

$$H' = \left[\begin{array}{cc} 0 & h \\ h & 0 \end{array} \right] \tag{5}$$

with

$$h = \frac{eB_x \hbar}{2m} \tag{6}$$

Using second order perturbation theory, the the perturbations on the two energies are

$$E_{a2} = \frac{h^2}{E_a - E_b} = -\frac{\hbar e B_x^2}{4mB_z} \tag{7}$$

$$E_{b2} = \frac{h^2}{E_b - E_a} = \frac{\hbar e B_x^2}{4mB_z} \tag{8}$$

The variational calculation gave the exact answer, which is

$$\langle H \rangle_{min} = \frac{1}{2}\left(E_a + E_b - \sqrt{(E_a - E_b)^2 + 4h^2}\right) \tag{9}$$

$$= \frac{\hbar e}{2m}\sqrt{B_x^2 + B_z^2} \tag{10}$$

That is, it is the same as 3 except we now have the magnitude of the full magnetic field.