

## WKB APPROXIMATION - ALTERNATIVE DERIVATION

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 8.2.

Here's an alternative derivation of the WKB approximation. This time we start with the wave function in the form

$$\psi(x) = e^{if(x)/\hbar} \quad (1)$$

where  $f(x)$  is now a complex function. Note that we don't need a separate amplitude term multiplying the exponential, since the imaginary part of  $f$  gives a real function:

$$e^{if(x)/\hbar} = e^{-Im(f(x))/\hbar} e^{iRe(f(x))/\hbar} \quad (2)$$

We can plug this into the one-dimensional time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x)\psi = E\psi \quad (3)$$

and rewrite it as

$$\frac{d^2\psi}{dx^2} = -\frac{2m[E - V(x)]}{\hbar^2}\psi \quad (4)$$

$$\equiv -\frac{p^2}{\hbar^2}\psi \quad (5)$$

where  $p = \sqrt{2m[E - V(x)]}$  is the classical formula for the momentum of a particle with total energy  $E$  moving in a one-dimensional potential  $V(x)$ , provided we assume  $E \geq V(x)$  for all  $x$ . The derivatives are

$$\psi' = e^{if/\hbar} \frac{if'}{\hbar} \quad (6)$$

$$= \frac{if'}{\hbar} \psi \quad (7)$$

$$\psi'' = \frac{if''}{\hbar} \psi + \frac{if'}{\hbar} \psi' \quad (8)$$

$$= \frac{if''}{\hbar} \psi - \left(\frac{f'}{\hbar}\right)^2 \psi \quad (9)$$

Putting this into 5 we get

$$\frac{if''}{\hbar} \psi - \left(\frac{f'}{\hbar}\right)^2 \psi = -\frac{p^2}{\hbar^2} \psi \quad (10)$$

$$i\hbar f'' - (f')^2 + p^2 = 0 \quad (11)$$

Since  $p$  can be an arbitrary function of  $x$ , we can't usually solve this differential equation in closed form, but we can try a series solution. If we expand  $f(x)$  in powers of  $\hbar$  we have

$$f(x) = \sum_{i=0}^{\infty} \hbar^i f_i(x) \quad (12)$$

so we can substitute this into 11 and require the coefficient of each power of  $\hbar$  to be zero:

$$i \sum_{i=1}^{\infty} \hbar^i f''_{i-1} - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \hbar^{i+j} f'_i f'_j + p^2 = 0 \quad (13)$$

$$-(f'_0)^2 + p^2 = 0 \quad (14)$$

$$if''_0 - 2f'_0 f'_1 = 0 \quad (15)$$

$$if''_1 - 2f'_0 f'_2 - (f'_1)^2 = 0 \quad (16)$$

Equation 14 can be solved as

$$f_0(x) = \pm \int p(x) dx \quad (17)$$

$$f''_0 = \pm p'(x) \quad (18)$$

Then from 15 we get

$$\pm ip' = \pm 2pf_1' \quad (19)$$

$$\frac{ip'}{2p} = f_1' \quad (20)$$

$$\frac{i}{2} \ln p = f_1 + \ln C \quad (21)$$

$$if_1 = -\frac{1}{2} \ln p + \ln C \quad (22)$$

To first order in  $\hbar$  we then get for the wave function:

$$\psi(x) = e^{i(f_0 + \hbar f_1)/\hbar} \quad (23)$$

$$= e^{if_1} e^{if_0/\hbar} \quad (24)$$

$$= \frac{C}{\sqrt{p}} e^{\pm \frac{i}{\hbar} \int p(x) dx} \quad (25)$$

which is the same formula for the WKB approximation that we got before.