WKB APPROXIMATION - ALTERNATIVE DERIVATION

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 8.2.

Here's an alternative derivation of the WKB approximation. This time we start with the wave function in the form

$$\psi(x)=e^{if(x)/\hbar} \tag{1}$$

where $f(x)$ is now a complex function. Note that we don't need a separate amplitude term multiplying the exponential, since the imaginary part of f gives a real function:

$$e^{if(x)/\hbar}=e^{-Im(f(x))/\hbar}e^{iRe(f(x))/\hbar} \tag{2}$$

We can plug this into the one-dimensional time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+V(x)\psi=E\psi \tag{3}$$

and rewrite it as

$$\begin{aligned}\frac{d^2\psi}{dx^2} &= -\frac{2m[E-V(x)]}{\hbar^2}\psi &(4)\\ &\equiv -\frac{p^2}{\hbar^2}\psi &(5)\end{aligned}$$

where $p=\sqrt{2m[E-V(x)]}$ is the classical formula for the momentum of a particle with total energy E moving in a one-dimensional potential $V(x)$, provided we assume $E\geq V(x)$ for all x.The derivatives are

$$\psi' = e^{if/\hbar}\frac{if'}{\hbar} \tag{6}$$

$$= \frac{if'}{\hbar}\psi \tag{7}$$

$$\psi'' = \frac{if''}{\hbar}\psi + \frac{if'}{\hbar}\psi' \tag{8}$$

$$= \frac{if''}{\hbar}\psi - \left(\frac{f'}{\hbar}\right)^2\psi \tag{9}$$

Putting this into 5 we get

$$\frac{if''}{\hbar}\psi - \left(\frac{f'}{\hbar}\right)^2\psi = -\frac{p^2}{\hbar^2}\psi \tag{10}$$

$$i\hbar f'' - \left(f'\right)^2 + p^2 = 0 \tag{11}$$

Since p can be an arbitrary function of x, we can't usually solve this differential equation in closed form, but we can try a series solution. If we expand $f(x)$ in powers of $\hbar$ we have

$$f(x) = \sum_{i=0}^{\infty}\hbar^i f_i(x) \tag{12}$$

so we can substitute this into 11 and require the coefficient of each power of $\hbar$ to be zero:

$$i\sum_{i=1}^{\infty}\hbar^i f''_{i-1} - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\hbar^{i+j}f'_i f'_j + p^2 = 0 \tag{13}$$

$$-\left(f'_0\right)^2 + p^2 = 0 \tag{14}$$

$$if''_0 - 2f'_0 f'_1 = 0 \tag{15}$$

$$if''_1 - 2f'_0 f'_2 - \left(f'_1\right)^2 = 0 \tag{16}$$

Equation 14 can be solved as

$$f_0(x) = \pm\int p(x)\,dx \tag{17}$$

$$f''_0 = \pm p'(x) \tag{18}$$

Then from 15 we get

$$\begin{aligned}
\pm ip' &= \pm 2pf_1' && (19)\\
\frac{ip'}{2p} &= f_1' && (20)\\
\frac{i}{2}\ln p &= f_1+\ln C && (21)\\
if_1 &= -\frac{1}{2}\ln p+\ln C && (22)
\end{aligned}$$

To first order in $\hbar$ we then get for the wave function:

$$\begin{aligned}
\psi(x) &= e^{i(f_0+\hbar f_1)/\hbar} && (23)\\
&= e^{if_1}e^{if_0/\hbar} && (24)\\
&= \frac{C}{\sqrt{p}}e^{\pm\frac{i}{\hbar}\int p(x)dx} && (25)
\end{aligned}$$

which is the same formula for the WKB approximation that we got before.