WKB APPROXIMATION: TUNNELING

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 8.3.

The WKB approximation can also be applied to cases where the particle's energy is less than the potential: $E < V(x)$. In these cases, we're looking at quantum tunneling. The derivation is similar to the previous case. We start with the Schrödinger equation in the form

$$\frac{d^2\psi}{dx^2} = -\frac{p^2}{\hbar^2}\psi \tag{0.1}$$

where $p = \sqrt{2m[E - V(x)]}$, although this time, since $E < V(x)$, p is purely imaginary, and $p^2 < 0$. If we follow through the alternative derivation (where we wrote the wave function in the form $\psi(x) = e^{if(x)/\hbar}$ and expanded $f(x)$ in powers of $\hbar$ so $f(x) = \sum_{i=0}^{\infty}\hbar^i f_i(x)$), we get as far as equation 17 in the earlier derivation:

$$f_0(x) = \pm\int p(x)\,dx \tag{0.2}$$

If we take the imaginary part of p to be positive, then we can write this as

$$f_0(x) = \pm i\int |p(x)|\,dx \tag{0.3}$$

Likewise, equation 20 in the earlier derivation is

$$\frac{ip'}{2p} = f_1' \tag{0.4}$$

Since p is purely imaginary, we can write this as

$$\frac{i|p|'}{2|p|} = f_1' \tag{0.5}$$

which has the solution

$$if_1 = -\frac{1}{2}\ln|p| + \ln C \tag{0.6}$$

Putting this together, we end up with the WKB approximation for tunneling as

$$\psi(x) \approx \frac{C}{\sqrt{|p|}} e^{\pm\frac{1}{\hbar}\int |p(x)|dx} \tag{0.7}$$

The wave function in this case is real (the exponent is real).

We can use this to analyze cases of transmission through a finite barrier, such as the finite square barrier we studied earlier. If we have a particle travelling in from left to right and it encounters a finite barrier from $x = -a$ to $x = a$, then we expect a reflected wave for $x < -a$ and a transmitted wave for $x > a$. Within the barrier, we can use the WKB approximation 0.7, so we have the wave function

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < -a \\ \frac{C}{\sqrt{|p|}} e^{\frac{1}{\hbar}\int |p(x)|dx} + \frac{D}{\sqrt{|p|}} e^{-\frac{1}{\hbar}\int |p(x)|dx} & -a < x < a \\ Fe^{ikx} & x > a \end{cases} \tag{0.8}$$

For a high and/or broad barrier, we'd expect the exponential decay term (with constant D) to dominate in the region $-a < x < a$. The transmission probability is defined as

$$T = \frac{|F|^2}{|A|^2} \tag{0.9}$$

and if we take the exponential decay as an estimate of the ratio of the amplitudes F and A, we get

$$\frac{|F|}{|A|} \sim e^{-\frac{1}{\hbar}\int_{-a}^{a}|p(x)|dx} \equiv e^{-\gamma} \tag{0.10}$$

so

$$T \approx e^{-2\gamma} \tag{0.11}$$

In the case of the finite square barrier with height $V_0 > E$ we have

$$p(x) = i\sqrt{2m(V_0-E)} \tag{0.12}$$

$$\gamma = \frac{1}{\hbar}\int_{-a}^{a}|p(x)|\,dx \tag{0.13}$$

$$= \frac{2a}{\hbar}\sqrt{2m(V_0-E)} \tag{0.14}$$

The exact result is given by equation 22 in the earlier post

$$T^{-1} = 1+\frac{V_0^2}{4E(V_0-E)}\sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(V_0-E)}\right) \tag{0.15}$$

$$= 1+\frac{V_0^2}{4E(V_0-E)}\sinh^2\gamma \tag{0.16}$$

For small transition probabilities, γ is large, so

$$\sinh\gamma = \frac{1}{2}\left(e^{\gamma}-e^{-\gamma}\right) \tag{0.17}$$

$$\approx \frac{e^{\gamma}}{2} \tag{0.18}$$

$$T^{-1} \approx \frac{V_0^2}{16E(V_0-E)}e^{2\gamma} \tag{0.19}$$

$$T \approx \frac{16E(V_0-E)}{V_0^2}e^{-2\gamma} \tag{0.20}$$

This isn't exactly the same as the WKB result 0.11, but the dependence on the barrier width ($2a$) is the same in both cases as it is contained in the exponential factor.

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