

## WKB APPROXIMATION: TUNNELING

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 8.3.

The WKB approximation can also be applied to cases where the particle's energy is less than the potential:  $E < V(x)$ . In these cases, we're looking at quantum tunneling. The derivation is similar to the previous case. We start with the Schrödinger equation in the form

$$\frac{d^2\psi}{dx^2} = -\frac{p^2}{\hbar^2}\psi \quad (1)$$

where  $p = \sqrt{2m[E - V(x)]}$ , although this time, since  $E < V(x)$ ,  $p$  is purely imaginary, and  $p^2 < 0$ . If we follow through the alternative derivation (where we wrote the wave function in the form  $\psi(x) = e^{if(x)/\hbar}$  and expanded  $f(x)$  in powers of  $\hbar$  so  $f(x) = \sum_{i=0}^{\infty} \hbar^i f_i(x)$ ), we get as far as equation 17 in the earlier derivation:

$$f_0(x) = \pm \int p(x) dx \quad (2)$$

If we take the imaginary part of  $p$  to be positive, then we can write this as

$$f_0(x) = \pm i \int |p(x)| dx \quad (3)$$

Likewise, equation 20 in the earlier derivation is

$$\frac{ip'}{2p} = f_1' \quad (4)$$

Since  $p$  is purely imaginary, we can write this as

$$\frac{i|p'|}{2|p|} = f_1' \quad (5)$$

which has the solution

$$if_1 = -\frac{1}{2} \ln |p| + \ln C \quad (6)$$

Putting this together, we end up with the WKB approximation for tunneling as

$$\psi(x) \approx \frac{C}{\sqrt{|p|}} e^{\pm \frac{1}{\hbar} \int |p(x)| dx} \quad (7)$$

The wave function in this case is real (the exponent is real).

We can use this to analyze cases of transmission through a finite barrier, such as the finite square barrier we studied earlier. If we have a particle travelling in from left to right and it encounters a finite barrier from  $x = -a$  to  $x = a$ , then we expect a reflected wave for  $x < -a$  and a transmitted wave for  $x > a$ . Within the barrier, we can use the WKB approximation 7, so we have the wave function

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < -a \\ \frac{C}{\sqrt{|p|}} e^{\frac{1}{\hbar} \int |p(x)| dx} + \frac{D}{\sqrt{|p|}} e^{-\frac{1}{\hbar} \int |p(x)| dx} & -a < x < a \\ Fe^{ikx} & x > a \end{cases} \quad (8)$$

For a high and/or broad barrier, we'd expect the exponential decay term (with constant  $D$ ) to dominate in the region  $-a < x < a$ . The transmission probability is defined as

$$T = \frac{|F|^2}{|A|^2} \quad (9)$$

and if we take the exponential decay as an estimate of the ratio of the amplitudes  $F$  and  $A$ , we get

$$\frac{|F|}{|A|} \sim e^{-\frac{1}{\hbar} \int_{-a}^a |p(x)| dx} \equiv e^{-\gamma} \quad (10)$$

so

$$T \approx e^{-2\gamma} \quad (11)$$

In the case of the finite square barrier with height  $V_0 > E$  we have

$$p(x) = i\sqrt{2m(V_0 - E)} \quad (12)$$

$$\gamma = \frac{1}{\hbar} \int_{-a}^a |p(x)| dx \quad (13)$$

$$= \frac{2a}{\hbar} \sqrt{2m(V_0 - E)} \quad (14)$$

The exact result is given by equation 22 in the earlier post

$$T^{-1} = 1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2 \left( \frac{2a}{\hbar} \sqrt{2m(V_0 - E)} \right) \quad (15)$$

$$= 1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2 \gamma \quad (16)$$

For small transition probabilities,  $\gamma$  is large, so

$$\sinh \gamma = \frac{1}{2} (e^\gamma - e^{-\gamma}) \quad (17)$$

$$\approx \frac{e^\gamma}{2} \quad (18)$$

$$T^{-1} \approx \frac{V_0^2}{16E(V_0 - E)} e^{2\gamma} \quad (19)$$

$$T \approx \frac{16E(V_0 - E)}{V_0^2} e^{-2\gamma} \quad (20)$$

This isn't exactly the same as the WKB result 11, but the dependence on the barrier width ( $2a$ ) is the same in both cases as it is contained in the exponential factor.

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