ALPHA DECAY USING THE WKB APPROXIMATION

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 8.4.

One of the early applications of the WKB approximation was George Gamov's 1928 theory of alpha decay. In a large nucleus, the nucleons (protons and neutrons) are held together by the strong nuclear force which at short range is stronger than the electric repulsion between the protons. The strong nuclear force is very short range, however, so if some nucleons can tunnel through the potential barrier, the electric force rapidly takes over resulting in the nucleons being ejected from the nucleus.

A common mode of decay is the emission of an alpha particle, consisting of 2 neutrons and 2 protons (a helium-4 nucleus). Gamow's model proposed that the nuclear force be modelled as a square well at potential $-V_0$ and width r_1 followed by the Coulomb repulsive potential for $r > r_1$. That is

$$V(x) = \begin{cases} -V_0 & 0 < r < r_1 \\ \frac{2Ze^2}{4\pi\varepsilon_0 r} & r > r_1 \end{cases} \tag{0.1}$$

where Z is the number of protons in the nucleus of the atom remaining after the emission of the alpha particle. The 2 in the numerator is the number of protons in the alpha particle.

If the alpha particle has an energy E such that $0 < E < \frac{2Ze^2}{4\pi\varepsilon_0 r_1^2}$ then its energy is below the maximum of V which occurs at $r = r_1$, so in order for it to escape, it must tunnel through the potential in the region $r_1 < r < r_2$ where the outer radius r_2 is determined by

$$E = \frac{2Ze^2}{4\pi\varepsilon_0 r_2} \tag{0.2}$$

$$r_2 = \frac{2Ze^2}{4\pi\varepsilon_0 E} \tag{0.3}$$

We saw that applying the WKB approximation to tunneling gave us a transmission probability of

$$T \approx e^{-2\gamma} \tag{0.4}$$

where

$$\gamma = \frac{1}{\hbar}\int_{r_1}^{r_2}\sqrt{2m\left(V\left(r\right)-E\right)}dr \tag{0.5}$$

$$= \frac{\sqrt{2m}}{\hbar}\int_{r_1}^{r_2}\sqrt{\frac{2Ze^2}{4\pi\varepsilon_0}}\sqrt{\frac{1}{r}-\frac{1}{r_2}}dr \tag{0.6}$$

$$= \frac{\sqrt{2m}}{\hbar}\sqrt{\frac{2Ze^2}{4\pi\varepsilon_0 r_2}}\int_{r_1}^{r_2}\sqrt{\frac{r_2}{r}-1}dr \tag{0.7}$$

$$= \frac{\sqrt{2mE}}{\hbar}\int_{r_1}^{r_2}\sqrt{\frac{r_2}{r}-1}dr \tag{0.8}$$

Doing the integral with Maple gives

$$\gamma = \frac{\sqrt{2mE}}{\hbar}\left[\frac{\pi}{4}r_2-\sqrt{r_1\left(r_2-r_1\right)}-\frac{r_2}{2}\arcsin\left(\frac{2r_1}{r_2}-1\right)\right] \tag{0.9}$$

We can write this in terms of $\rho \equiv r_1/r_2$:

$$\gamma = \frac{\sqrt{2mE}}{\hbar}\frac{r_2}{4}\left(\pi-4\sqrt{\rho\left(1-\rho\right)}-2\arcsin\left(2\rho-1\right)\right) \tag{0.10}$$

If we assume that $r_1 \ll r_2$ (that is, the range of the nuclear force is much less than the point where the alpha particle breaks free of the nucleus, which is usually the case), then a series expansion about $\rho = 0$ gives

$$\gamma = \frac{\sqrt{2mE}}{\hbar}\left[\frac{\pi}{2}r_2-2r_2\sqrt{\rho}+\frac{r_2}{3}\rho^{3/2}+\ldots\right] \tag{0.11}$$

$$= \frac{\sqrt{2mE}}{\hbar}\left[\frac{\pi}{2}r_2-2\sqrt{r_1r_2}+\frac{r_1}{3}\sqrt{\frac{r_1}{r_2}}+\ldots\right] \tag{0.12}$$

Taking just the first two terms, we have, using 0.3

$$\gamma \cong \frac{\sqrt{2mE}}{\hbar}\frac{Ze^2}{4\varepsilon_0 E} - \frac{2\sqrt{2mE}}{\hbar}\sqrt{\frac{2Ze^2}{4\pi\varepsilon_0 E}}\sqrt{r_1} \tag{0.13}$$

$$= \frac{e^2\pi\sqrt{2m}}{4\pi\varepsilon_0\hbar}\frac{Z}{\sqrt{E}} - \frac{4\sqrt{m}}{\hbar}\sqrt{\frac{e^2}{4\pi\varepsilon_0}}\sqrt{Zr_1} \tag{0.14}$$

$$= K_1\frac{Z}{\sqrt{E}} - K_2\sqrt{Zr_1} \tag{0.15}$$

where the constants are

$$K_1 = \frac{e^2\pi\sqrt{2m}}{4\pi\varepsilon_0\hbar} \tag{0.16}$$

$$K_2 = \frac{4\sqrt{m}}{\hbar}\sqrt{\frac{e^2}{4\pi\varepsilon_0}} \tag{0.17}$$

Plugging in the numbers gives (in SI units; remember that m is the mass of the emitted alpha particle):

$$\frac{e^2}{4\pi\varepsilon_0} = 2.307\times10^{-28}\text{m}^3\text{kg s}^{-2} \tag{0.18}$$

$$\hbar = 1.0546\times10^{-34}\text{ kg m}^2\text{ s}^{-1} \tag{0.19}$$

$$m = 6.645\times10^{-27}\text{ kg} \tag{0.20}$$

$$K_1 = 7.923\times10^{-7}\text{ kg}^{1/2}\text{m s}^{-1} \tag{0.21}$$

$$= 7.923\times10^{-7}\text{ J}^{1/2} \tag{0.22}$$

$$= 1.979\text{ MeV}^{1/2} \tag{0.23}$$

$$K_2 = 4.696\times10^{7}\text{ m}^{-1/2} \tag{0.24}$$

$$= 1.485\text{ fm}^{-1/2} \tag{0.25}$$

where the fm is a *fermi* or 10^{-15} m, so $10^{15}\text{ m}^{-1} = 1\text{ fm}^{-1}$ and $\sqrt{10^{15}}\text{ m}^{-1/2} = 3.16\times10^{7}\text{ m}^{-1/2} = 1\text{ fm}^{-1/2}$.

Example 1. As a specific example, we'll look at ^{238}U. When this isotope of uranium emits an alpha particle, the remaining nucleus is ^{234}Th (thorium). The masses can be found at this site, and we get

$$^{238}\mathrm{U}: \ 238.050782 \tag{0.26}$$
$$^{234}\mathrm{Th}: \ 234.043595 \tag{0.27}$$
$$^{4}\mathrm{He}: \ 4.002603 \tag{0.28}$$

The masses are in unified atomic mass units (u), where one u is 1/12 the mass of ^{12}C or 1.66×10^{-27} kg or 931.5 MeV.

Returning to ^{238}U we need values for r_1 and E. From experiment, we have

$$r_1 \cong 1.07 A^{1/3} \text{ fm} \tag{0.29}$$

where A is the total number of protons and neutrons in the nucleus. From special relativity, we can get the energy of the emitted alpha particle as the difference between the rest masses of the parent nucleus m_p and those of the residual nucleus m_r plus alpha particle m_α:

$$E = m_p c^2 - \left(m_r c^2 + m_\alpha c^2\right) \tag{0.30}$$

From the numbers above, we get

$$r_1 = 1.07\,(238)^{1/3} = 6.63 \text{ fm} \tag{0.31}$$
$$E = (238.050782 - 234.043595 - 4.002603) \times 931.5 \tag{0.32}$$
$$= 4.27 \text{ MeV} \tag{0.33}$$

To calculate γ we use 0.15 with $Z = 90$ for the decay product (thorium), and we get

$$\gamma = 50.06 \tag{0.34}$$

To get a prediction of the lifetime of a ^{238}U nucleus, we can use a crude argument to get a rough estimate. Suppose that, before its escape, the alpha particle is moving within the nucleus the some velocity v, giving it a kinetic energy of

$$E_k = \frac{1}{2} m_\alpha v^2 \tag{0.35}$$

If we're interested only in an order of magnitude estimate, we can ignore the nuclear binding energy $-V_0$ within the nucleus and take $E \approx E_k$. (In reality, $E = E_k - V_0$, so making this assumption underestimates the velocity.) On average, the alpha particle will collide with the boundary (that is, it will reach $r = r_1$) with a frequency of $v/2r_1$ times per second (since it travels the diameter of the nucleus between collisions), and from 0.4, the probability of tunnelling through the barrier is $T \approx e^{-2\gamma}$ on each collision. Thus the probability of emission per second is about $ve^{-2\gamma}/2r_1$ making the approximate lifetime τ of the parent nucleus equal to the reciprocal of this or

$$\tau \approx \frac{2r_1}{v} e^{2\gamma} \tag{0.36}$$

For ^{238}U we get

$$\begin{aligned}
v &= \sqrt{\frac{2E}{m_\alpha}} && (0.37)\\
&= \sqrt{\frac{2E}{m_\alpha c^2}}c && (0.38)\\
&= \sqrt{\frac{2\times 4.27}{4.002603\times 931.5}}c && (0.39)\\
&= 1.436\times 10^7 \text{ m s}^{-1} && (0.40)
\end{aligned}$$

This is about 4% of the speed of light, so we're justified in using a non-relativistic approximation. The lifetime is thus

$$\tau = 2.8\times 10^{22} \text{ s} = 8.87\times 10^{14} \text{ years} \tag{0.41}$$

The actual half-life of ^{238}U is around 4.5×10^9 years, so this estimate is pretty wide of the mark, but at least it agrees that the lifetime is very long.

Example 2. We can do the same calculation for ^{212}Po which decays to ^{208}Pb and we find the atomic masses are

$$\begin{aligned}
{}^{212}\text{Po}: &\ 211.988851\\
{}^{208}\text{Pb}: &\ 207.976635
\end{aligned}$$

Plugging in the numbers as before, we get

$$\begin{aligned}
r_1 &= 6.34 \text{ fm} && (0.42)\\
E &= 8.954 \text{ MeV} && (0.43)\\
v &= 2.079\times 10^7 \text{ m s}^{-1} && (0.44)\\
\gamma &= 20.399 && (0.45)\\
\tau &= 3.19\times 10^{-4} \text{ s} && (0.46)
\end{aligned}$$

The experimental value is 3×10^{-7} s so again the WKB calculation is quite a bit longer, but at least it gives good qualitative agreement. Because the dependence on γ (and thus on E, Z and r_1) is exponential, relatively small changes in these quantities translate into a huge difference in lifetime.

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