AIRY FUNCTIONS AND THE BOUNCING ELECTRON

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 8.5.

In order to progress further with our study of the WKB approximation we'll need to study a differential equation known as *Airy's equation*, named after George Biddell Airy (1801 - 1892), an English mathematician and astronomer (he served as Astronomer Royal for many years). We'll leave the application to the WKB approximation for the next post; here we'll look at how Airy's equation arises in quantum mechanics and do an example of its solution.

Suppose we have a particle of mass m that moves vertically under the influence of gravity at the Earth's surface, and that it bounces elastically off some hard surface which occupies the plane $x=0$. The potential is

$$V(x)=\begin{cases}\infty & x<0\\ mgx & x>0\end{cases} \tag{1}$$

The Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+mgx\psi=E\psi \tag{2}$$

$$\frac{\partial^2\psi}{\partial x^2}=\frac{2m^2g}{\hbar^2}\left(x-\frac{E}{mg}\right)\psi \tag{3}$$

$$\equiv\alpha^3\left(x-\frac{E}{mg}\right)\psi \tag{4}$$

$$\alpha\equiv\left(\frac{2m^2g}{\hbar^2}\right)^{1/3} \tag{5}$$

Now we use the substitution

$$z \equiv \alpha\left(x-\frac{E}{mg}\right) \tag{6}$$

$$\frac{\partial^2\psi}{\partial x^2} = \alpha^2\frac{\partial^2\psi}{\partial z^2} \tag{7}$$

$$= \alpha^3\frac{z}{\alpha}\psi \tag{8}$$

$$\frac{\partial^2\psi}{\partial z^2} = z\psi \tag{9}$$

Equation 9 is Airy's equation. Its solutions are called Airy functions. As this is a second order differential equation, there are two linearly independent solutions, known as $Ai(z)$ and $Bi(z)$. They are not easily expressed in terms of anything simpler, but they do have some useful properties that aren't too hard to state.

Both $Ai(z)$ and $Bi(z)$ oscillate about the z axis for $z<0$, so there are a number of zeroes for these functions in that region. As $z\to+\infty$, $Ai(z)\to 0$ and $Bi(z)\to\infty$. Their asymptotic forms are, for reference:

$$Ai(z)\sim\begin{cases}\frac{1}{\sqrt{\pi}(-z)^{1/4}}\sin\left[\frac{2}{3}(-z)^{3/2}+\frac{\pi}{4}\right] & z\ll 0\\ \frac{1}{2\sqrt{\pi}z^{1/4}}e^{-2z^{3/2}/3} & z\gg 0\end{cases} \tag{10}$$

$$Bi(z)\sim\begin{cases}\frac{1}{\sqrt{\pi}(-z)^{1/4}}\cos\left[\frac{2}{3}(-z)^{3/2}+\frac{\pi}{4}\right] & z\ll 0\\ \frac{1}{\sqrt{\pi}z^{1/4}}e^{2z^{3/2}/3} & z\gg 0\end{cases} \tag{11}$$

For our example, we can exclude $Bi(z)$ since it blows up as z gets large, so the general solution is

$$\psi(z)=\begin{cases}0 & z<-\frac{E}{mg}\\ aAi(z) & z>\frac{E}{mg}\end{cases} \tag{12}$$

or, in terms of x:

$$\psi\left(\alpha\left(x-\frac{E}{mg}\right)\right)=\begin{cases}0 & x<0\\ aAi\left(\alpha\left(x-\frac{E}{mg}\right)\right) & x>0\end{cases} \tag{13}$$

To find the allowed energies, we impose the boundary condition that $\psi(x)$ is continuous at $x=0$, which gives us

$$Ai\left(-\frac{\alpha E}{mg}\right)=0 \tag{14}$$

To solve this, we need the zeroes of $Ai(x)$ which can be found from tables, or by using Maple's fsolve method on the AiryAi function. For $g=9.8\text{ m s}^{-2}$ and a mass of 0.1 kgwe have

$$\begin{aligned}
\alpha &= 2.602\times10^{22}\text{ m}^{-1} && (15)\\
E_1 &= 8.805\times10^{-23}\text{ J} && (16)\\
E_2 &= 1.539\times10^{-22}\text{ J} && (17)\\
E_3 &= 2.079\times10^{-22}\text{ J} && (18)\\
E_4 &= 2.556\times10^{-22}\text{ J} && (19)
\end{aligned}$$

If the particle is an electron, $m=9.11\times10^{-31}$ kg and we get

$$\begin{aligned}
\alpha &= 1135\text{ m}^{-1} && (20)\\
E_1 &= 1.839\times10^{-32}\text{ J} && (21)\\
&= 1.148\times10^{-13}\text{ eV} && (22)
\end{aligned}$$

Using the virial theorem we can find the electron's average height above the floor in its ground state:

$$\begin{aligned}
2\langle T\rangle &= \left\langle x\frac{dV}{dx}\right\rangle && (23)\\
&= mg\langle x\rangle && (24)\\
&= \langle V\rangle && (25)\\
E_1 &= \langle T\rangle+\langle V\rangle && (26)\\
&= \frac{3}{2}\langle V\rangle && (27)\\
&= \frac{3}{2}mg\langle x\rangle && (28)\\
\langle x\rangle &= \frac{2E_1}{3mg} && (29)\\
&= 0.00137\text{ m} && (30)\\
&= 1.37\text{ mm} && (31)
\end{aligned}$$

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