WKB APPROXIMATION OF THE HARMONIC OSCILLATOR

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 8.7.

We've seen that if we apply the WKB approximation to a region where the potential is increasing we get

$$\psi(x) \approx \begin{cases} \frac{2D}{\sqrt{p(x)}} \sin\left[\int_x^{x_2} p\,dx'/\hbar + \pi/4\right] & x < x_2 \\ \frac{D}{\sqrt{|p(x)|}} \exp\left[-\int_{x_2}^{x} |p(x')|\,dx'/\hbar\right] & x > x_2 \end{cases} \tag{0.1}$$

where x_2 is the turning point, where the particle's energy $E = V(x_2)$ with $V'(x_2) > 0$ and D is a normalization constant.

At a point x_1 where the potential is decreasing and $E = V(x_1)$ with $V'(x_1) < 0$ we get

$$\psi(x) \approx \begin{cases} \frac{2D'}{\sqrt{p(x)}} \sin\left[\int_{x_1}^{x} p\,dx'/\hbar + \pi/4\right] & x > x_1 \\ \frac{D'}{\sqrt{|p(x)|}} \exp\left[-\int_{x}^{x_1} |p(x')|\,dx'/\hbar\right] & x < x_1 \end{cases} \tag{0.2}$$

where D' is another normalization constant.

If we're applying the WKB approximation to a potential well where the energy E is greater than $V(x)$ only in the region $x_1 < x < x_2$ then the wave function in this region can be written as the sine function from either of these two forms, that is

$$\psi(x) \approx \frac{2D}{\sqrt{p(x)}} \sin\left[\int_x^{x_2} p\,dx'/\hbar + \pi/4\right] \tag{0.3}$$

$$\approx \frac{2D'}{\sqrt{p(x)}} \sin\left[\int_{x_1}^{x} p\,dx'/\hbar + \pi/4\right] \tag{0.4}$$

Because the sine is an odd function, we can write the second form as

$$\psi(x) \approx -\frac{2D'}{\sqrt{p(x)}} \sin\left[-\int_{x_1}^{x} p\,dx'/\hbar - \pi/4\right] \tag{0.5}$$

The zeroes of the sines must match up between these two forms which means the arguments of these two sines must be equal up to a multiple of π:

$$\frac{1}{\hbar}\int_{x}^{x_2} p\,dx' + \frac{\pi}{4} = -\frac{1}{\hbar}\int_{x_1}^{x} p\,dx' - \frac{\pi}{4} + n\pi \tag{0.6}$$

$$\left(\int_{x_1}^{x} + \int_{x}^{x_2}\right) p\,dx' = \left(n - \frac{1}{2}\right)\pi\hbar \tag{0.7}$$

$$\int_{x_1}^{x_2} p\,dx = \left(n - \frac{1}{2}\right)\pi\hbar \tag{0.8}$$

where $n = 1, 2, 3, \ldots$ (we have to start at $n = 1$ rather than $n = 0$ to keep the integral positive).

We can make the two approximate forms of ψ equal if we also set $D' = -D$.

Example. As a simple illustration of this, we consider the harmonic oscillator, with a potential

$$V(x) = \frac{1}{2}kx^2 \tag{0.9}$$

$$p(x) = \sqrt{2m\left(E - \frac{1}{2}kx^2\right)} \tag{0.10}$$

In this case, the turning points are

$$x_1 = -\sqrt{\frac{2E}{k}} \tag{0.11}$$

$$x_2 = \sqrt{\frac{2E}{k}} \tag{0.12}$$

and

$$\int_{x_1}^{x_2} p\,dx = \sqrt{2m}\int_{-\sqrt{2E/k}}^{\sqrt{2E/k}}\sqrt{E-\frac{kx^2}{2}}dx \tag{0.13}$$

$$= \pi E\sqrt{\frac{m}{k}} \tag{0.14}$$

$$= \left(n-\frac{1}{2}\right)\pi\hbar \tag{0.15}$$

$$E = \left(n-\frac{1}{2}\right)\sqrt{\frac{k}{m}}\hbar \tag{0.16}$$

$$= \left(n-\frac{1}{2}\right)\hbar\omega \tag{0.17}$$

Since n starts at 1, this gives the same sequence of energies as the exact analysis.

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