

WKB APPROXIMATION AT A TURNING POINT WITH DECREASING POTENTIAL

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 8.9.

In the last post, we worked out the WKB approximation on either side of a turning point where the potential is increasing. We can do the same analysis for a turning point where the potential is decreasing. The calculations are almost exactly the same as in the first case.

The wave function has the form

$$(1) \quad \psi(x) \approx \frac{1}{\sqrt{p(x)}} \left(B e^{i \int p dx / \hbar} + C e^{-i \int p dx / \hbar} \right)$$

if the particle's energy $E > V(x)$, where

$$(2) \quad p(x) = \sqrt{2m(E - V(x))}$$

is the momentum of the particle. In the tunneling case, where $E < V(x)$, the WKB approximation gives

$$(3) \quad \psi(x) \approx \frac{1}{\sqrt{|p|}} \left(D e^{-\frac{1}{\hbar} \int |p(x)| dx} + F e^{\frac{1}{\hbar} \int |p(x)| dx} \right)$$

In the decreasing potential case, we have $E < V(x)$ for $x < x_1$ and $E > V(x)$ for $x > x_1$. As before, we'll move the turning point so that $x_1 = 0$ and shift it back after the analysis. In this case, the limits on the integrals above give the results

$$(4) \quad \psi(x) = \begin{cases} \frac{1}{\sqrt{|p|}} \left(D e^{-\frac{1}{\hbar} \int_x^0 |p(x')| dx'} + F e^{\frac{1}{\hbar} \int_x^0 |p(x')| dx'} \right) & x < 0 \\ \frac{1}{\sqrt{p(x)}} \left(B e^{i \int_0^x p(x') dx' / \hbar} + C e^{-i \int_0^x p(x') dx' / \hbar} \right) & x > 0 \end{cases}$$

As the integrand for $x < 0$ is non-negative, we must have $F = 0$ to keep ψ finite as $x \rightarrow -\infty$, so we have

$$(5) \quad \psi(x) = \begin{cases} \frac{1}{\sqrt{|p|}} \left(D e^{-\frac{1}{\hbar} \int_x^0 |p(x')| dx'} \right) & x < 0 \\ \frac{1}{\sqrt{p(x)}} \left(B e^{i \int_0^x p(x') dx' / \hbar} + C e^{-i \int_0^x p(x') dx' / \hbar} \right) & x > 0 \end{cases}$$

As before, we assume that around $x = 0$

$$(6) \quad V(x) \approx E + V'(0)x$$

where now $V'(0) < 0$ since the potential is decreasing. We now define the patching function ψ_p that satisfies the Schrödinger equation with this approximate potential near $x = 0$.

$$(7) \quad \frac{d^2 \psi_p}{dx^2} = \frac{2mV'(0)}{\hbar^2} x \psi_p$$

$$(8) \quad = -\frac{2m|V'(0)|}{\hbar^2} x \psi_p$$

By defining

$$(9) \quad \alpha \equiv \left(\frac{2m|V'(0)|}{\hbar^2} \right)^{1/3}$$

$$(10) \quad z \equiv -\alpha x$$

we end up with Airy's equation

$$(11) \quad \frac{d^2 \psi_p}{dz^2} = z \psi_p$$

As we saw earlier, this has the general solution

$$(12) \quad \psi_p(z) = aAi(z) + bBi(z)$$

$$(13) \quad \psi_p(x) = aAi(-\alpha x) + bBi(-\alpha x)$$

It is this function that we need to match up to 5.

To do this, we make a further assumption that the overlap regions (where we're trying to match the WKB functions with ψ_p) are far enough from $x = 0$ that we can use the asymptotic forms of the Airy functions. These forms are

$$(14) \quad Ai(z) \sim \begin{cases} \frac{1}{\sqrt{\pi}(-z)^{1/4}} \sin \left[\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4} \right] & z \ll 0 \\ \frac{1}{2\sqrt{\pi z^{1/4}}} e^{-2z^{3/2}/3} & z \gg 0 \end{cases}$$

$$(15) \quad Bi(z) \sim \begin{cases} \frac{1}{\sqrt{\pi}(-z)^{1/4}} \cos \left[\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4} \right] & z \ll 0 \\ \frac{1}{\sqrt{\pi z^{1/4}}} e^{2z^{3/2}/3} & z \gg 0 \end{cases}$$

The forms can't be used if we get too close to $x = 0$, but the hope is that there is still some range of x where both the WKB wave function and ψ_p are reasonable approximations to the true wave function. Under this assumption, we have from 13 for $z = -\alpha x \gg 0$, or $x \ll 0$:

$$(16) \quad \psi_p(x) = \frac{a}{2\sqrt{\pi z^{1/4}}} e^{-2z^{3/2}/3} + \frac{b}{\sqrt{\pi z^{1/4}}} e^{2z^{3/2}/3}$$

$$(17) \quad = \frac{a}{2\sqrt{\pi}(-\alpha x)^{1/4}} e^{-2(-\alpha x)^{3/2}/3} + \frac{b}{\sqrt{\pi}(\alpha x)^{1/4}} e^{2(-\alpha x)^{3/2}/3}$$

With the potential 6, the WKB function 5 for $x < 0$ is

$$(18) \quad p(x) = \sqrt{2m(E - V(x))}$$

$$(19) \quad = \sqrt{-2mV'(0)x}$$

$$(20) \quad = \hbar\sqrt{x}\alpha^{3/2}$$

$$(21) \quad \psi(x) = \frac{1}{\sqrt{|p|}} D e^{-\frac{1}{\hbar} \int_x^0 |p(x')| dx'}$$

$$(22) \quad \int_0^x |p(x')| dx' = \sqrt{-2mV'(0)} \int_x^0 \sqrt{-x'} dx'$$

$$(23) \quad = \sqrt{2m|V'(0)|} \frac{2}{3} (-x)^{3/2}$$

$$(24) \quad = \frac{2}{3} \hbar (-\alpha x)^{3/2}$$

$$(25) \quad \psi(x) = \frac{D}{\sqrt{\hbar}\alpha^{3/4}(-x)^{1/4}} e^{-2(-\alpha x)^{3/2}/3}$$

Matching this up with 17, we see that $b = 0$ and

$$(26) \quad \frac{a}{2\sqrt{\pi}\alpha^{1/4}} = \frac{D}{\sqrt{\hbar}\alpha^{3/4}}$$

$$(27) \quad a = 2\sqrt{\frac{\pi}{\hbar}\alpha}D$$

The patching function must therefore have the form

$$(28) \quad \psi_p(x) = 2\sqrt{\frac{\pi}{\hbar}\alpha}De^{-2(-\alpha x)^{3/2}/3}$$

for $\alpha x \ll 0$.

We can do the same calculation for the overlap region for $x > 0$, where $E > V(x)$. Since $x > 0$, $z = -\alpha x < 0$ and we can use the asymptotic form of $Ai(x)$ for large negative x and we get

$$(29) \quad \psi_p(x) = \frac{a}{\sqrt{\pi}(\alpha x)^{1/4}} \sin \left[\frac{2}{3}(\alpha x)^{3/2} + \frac{\pi}{4} \right]$$

For the WKB function, we start with 5

$$(30) \quad p(x) = \sqrt{-2mV'(0)x}$$

$$(31) \quad = \hbar\alpha^{3/2}\sqrt{x}$$

$$(32) \quad \int_0^x p(x') dx' = \frac{2}{3}\hbar(\alpha x)^{3/2}$$

$$(33) \quad \psi(x) = \frac{1}{\sqrt{\hbar}\alpha^{3/4}(x)^{1/4}} \left(Be^{2i(\alpha x)^{3/2}/3} + Ce^{-2i(\alpha x)^{3/2}/3} \right)$$

We can now compare this with 29 by writing the sine in terms of exponentials

$$(34) \quad \psi_p(x) = \frac{a}{2i\sqrt{\pi}(\alpha x)^{1/4}} \left[e^{i\left[\frac{2}{3}(\alpha x)^{3/2} + \frac{\pi}{4}\right]} - e^{-i\left[\frac{2}{3}(\alpha x)^{3/2} + \frac{\pi}{4}\right]} \right]$$

Comparing terms, we have

$$(35) \quad \frac{B}{\sqrt{\hbar}\alpha^{3/4}} = \frac{ae^{i\pi/4}}{2i\sqrt{\pi}\alpha^{1/4}}$$

$$(36) \quad B = \frac{ae^{i\pi/4}\sqrt{\hbar}\alpha}{2i\sqrt{\pi}}$$

$$(37) \quad = -ie^{i\pi/4}D$$

$$(38) \quad \frac{C}{\sqrt{\hbar}\alpha^{3/4}} = -\frac{ae^{-i\pi/4}}{2i\sqrt{\pi}\alpha^{1/4}}$$

$$(39) \quad C = -\frac{ae^{-i\pi/4}\sqrt{\hbar}\alpha}{2i\sqrt{\pi}}$$

$$(40) \quad = ie^{-i\pi/4}D$$

We now have the constants B and C , belonging to the WKB wave function for $x > 0$, in terms of the constant D belonging to the WKB function for $x < 0$, which is what we were after. Note that the WKB functions are still valid only for values of x that maintain a respectable distance from the turning point; all we have done is connected the approximations on either side of the turning point. The final form of the WKB function is, for $x > 0$:

$$(41) \quad \psi(x) \cong \frac{-iD}{\sqrt{p(x)}} \left(e^{i(\int p dx/\hbar + \pi/4)} - e^{-i(\int p dx/\hbar + \pi/4)} \right)$$

$$(42) \quad = \frac{2D}{\sqrt{p(x)}} \frac{1}{2i} \left(e^{i(\int p dx/\hbar + \pi/4)} - e^{-i(\int p dx/\hbar + \pi/4)} \right)$$

$$(43) \quad = \frac{2D}{\sqrt{p(x)}} \sin \left[\int_0^x p dx'/\hbar + \pi/4 \right]$$

And for $x < 0$

$$(44) \quad \psi(x) \cong \frac{D}{\sqrt{|p(x)|}} \exp \left[-\int_x^0 |p(x')| dx'/\hbar \right]$$

If the turning point is at the general location of x_1 , we can just replace the 0 in the limits of the integrals by x_1 . The only difference between the cases of increasing and decreasing potentials is that the limits of integration are reversed.

PINGBACKS

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