WKB APPROXIMATION AT A TURNING POINT WITH DECREASING POTENTIAL

In the last post, we worked out the WKB approximation on either side of a turning point where the potential is increasing. We can do the same analysis for a turning point where the potential is decreasing. The calculations are almost exactly the same as in the first case.

The wave function has the form

$$\psi(x) \approx \frac{1}{\sqrt{|p(x)|}} \left( B e^{\frac{1}{\hbar} \int p(x) dx} + C e^{-\frac{1}{\hbar} \int p(x) dx} \right)$$

(1)

if the particle’s energy $E > V(x)$, where

$$p(x) = \sqrt{2m(E - V(x))}$$

(2)

is the momentum of the particle. In the tunneling case where $E < V(x)$, the WKB approximation gives

$$\psi(x) \approx \frac{1}{\sqrt{|p(x)|}} \left( D e^{\frac{1}{\hbar} \int |p(x)| dx} + F e^{\frac{1}{\hbar} \int |p(x)| dx} \right)$$

(3)

In the decreasing potential case, we have $E < V(x)$ for $x < x_1$ and $E > V(x)$ for $x > x_1$. As before, we’ll move the turning point so that $x_1 = 0$ and shift it back after the analysis. In this case, the limits on the integrals above give the results

$$\psi(x) = \begin{cases} 
\frac{1}{\sqrt{|p(x)|}} \left( D e^{\frac{1}{\hbar} \int_0^x |p(x')| dx'} + F e^{\frac{1}{\hbar} \int_0^x |p(x')| dx'} \right) & x < 0 \\
\frac{1}{\sqrt{p(x)}} \left( B e^{\frac{1}{\hbar} \int_0^x p(x') dx'/\hbar} + C e^{-\frac{1}{\hbar} \int_0^x p(x') dx'/\hbar} \right) & x > 0 
\end{cases}$$

(4)

As the integrand for $x < 0$ is non-negative, we must have $F = 0$ to keep $\psi$ finite as $x \to -\infty$, so we have
\[ \psi(x) = \begin{cases} \frac{1}{\sqrt{|p|}} \left( D e^{-\frac{i}{\hbar} \int_0^x p(x') dx'} \right) & x < 0 \\ \frac{1}{\sqrt{p(x)}} \left( B e^{i \int_0^x p(x') dx'/\hbar} + C e^{-i \int_0^x p(x') dx'/\hbar} \right) & x > 0 \end{cases} \]  

(5)

As before, we assume that around \( x = 0 \)

\[ V(x) \approx E + V'(0) x \]  

(6)

where now \( V'(0) < 0 \) since the potential is decreasing. We now define the patching function \( \psi_p \) that satisfies the Schrödinger equation with this approximate potential near \( x = 0 \).

\[ \frac{d^2 \psi_p}{dx^2} = \frac{2mV'(0)}{\hbar^2} \psi_p \]  

(7)

\[ = -\frac{2m|V'(0)|}{\hbar^2} \psi_p \]  

(8)

By defining

\[ \alpha \equiv \left( \frac{2m|V'(0)|}{\hbar^2} \right)^{1/3} \]  

(9)

\[ z \equiv -\alpha x \]  

(10)

we end up with **Airy’s equation**

\[ \frac{d^2 \psi_p}{dz^2} = z \psi_p \]  

(11)

As we saw earlier, this has the general solution

\[ \psi_p(z) = aAi(z) + bBi(z) \]  

(12)

\[ \psi_p(x) = aAi(-\alpha x) + bBi(-\alpha x) \]  

(13)

It is this function that we need to match up to [5].

To do this, we make a further assumption that the overlap regions (where we’re trying to match the WKB functions with \( \psi_p \)) are far enough from \( x = 0 \) that we can use the asymptotic forms of the Airy functions. These forms are
Ai (z) \sim \begin{cases} \frac{1}{\sqrt{\pi (-z)^{1/4}}} \sin \left[ \frac{2}{3} (-z)^{3/2} + \frac{\pi}{4} \right] & z \ll 0 \\ \frac{1}{2\sqrt{\pi z^{1/4}}} e^{-2z^{3/2}/3} & \text{otherwise} \end{cases}

(14)

Bi (z) \sim \begin{cases} \frac{1}{\sqrt{\pi (-z)^{1/4}}} \cos \left[ \frac{2}{3} (-z)^{3/2} + \frac{\pi}{4} \right] & z \ll 0 \\ \frac{1}{\sqrt{\pi z^{1/4}}} e^{2z^{3/2}/3} & \text{otherwise} \end{cases}

(15)

The forms can’t be used if we get too close to $x = 0$, but the hope is that there is still some range of $x$ where both the WKB wave function and $\psi_p$ are reasonable approximations to the true wave function. Under this assumption, we have from (13) for $z = -\alpha x \gg 0$, or $x \ll 0$:

$$\psi_p (x) = \frac{a}{2\sqrt{\pi z^{1/4}}} e^{-2z^{3/2}/3} + \frac{b}{\sqrt{\pi z^{1/4}}} e^{2z^{3/2}/3}$$

(16)

$$= \frac{a}{2\sqrt{\pi (-\alpha x)^{1/4}}} e^{-2(-\alpha x)^{3/2}/3} + \frac{b}{\sqrt{\pi (\alpha x)^{1/4}}} e^{2(-\alpha x)^{3/2}/3}$$

(17)

With the potential $6$, the WKB function $5$ for $x < 0$ is

$$p (x) = \sqrt{2m (E - V (x))}$$

(18)

$$= \sqrt{-2m V' (0)} x$$

(19)

$$= \hbar \sqrt{\pi} \alpha^{3/2}$$

(20)

$$\psi (x) = \frac{1}{\sqrt{|p|}} De^{-\frac{1}{2} \int_0^x |p(x')| dx'}$$

(21)

$$\int_0^x |p (x')| dx' = \sqrt{-2m V' (0)} \int_0^x \sqrt{-x'} dx'$$

(22)

$$= \sqrt{2m |V' (0)|} \frac{2}{3} (-x)^{3/2}$$

(23)

$$= \frac{2}{3} \hbar (-\alpha x)^{3/2}$$

(24)

$$\psi (x) = \frac{D}{\sqrt{\hbar} \alpha^{3/4} (-x)^{1/4}} e^{2(-\alpha x)^{3/2}/3}$$

(25)

Matching this up with (17), we see that $b = 0$ and
\[
\frac{a}{2\sqrt{\pi}\alpha^{1/4}} = \frac{D}{\sqrt{\hbar}\alpha^{3/4}} \tag{26}
\]
\[
a = 2\sqrt{\frac{\pi}{\hbar}}D \tag{27}
\]

The patching function must therefore have the form

\[
\psi_p(x) = 2\sqrt{\frac{\pi}{\hbar}}De^{-2(-\alpha x)^{3/2}/3} \tag{28}
\]

for \(\alpha x \ll 0\).

We can do the same calculation for the overlap region for \(x > 0\), where \(E > V(x)\). Since \(x > 0\), \(z = -\alpha x < 0\) and we can use the asymptotic form of \(Ai(x)\) for large negative \(x\) and we get

\[
\psi_p(x) = \frac{a}{\sqrt{\pi}(\alpha x)^{1/4}} \sin \left[ \frac{2}{3} (\alpha x)^{3/2} + \frac{\pi}{4} \right] \tag{29}
\]

For the WKB function, we start with [5]

\[
p(x) = \sqrt{-2mV'(0)}x = \hbar\alpha^{3/2}\sqrt{x} \tag{30}
\]
\[
\int_0^x p(x') \, dx' = \frac{2}{3} \hbar (\alpha x)^{3/2} \tag{31}
\]
\[
\psi(x) = \frac{1}{\sqrt{\hbar}\alpha^{3/4} (x)^{1/4}} \left( Be^{2i(\alpha x)^{3/2}/3} + Ce^{-2i(\alpha x)^{3/2}/3} \right) \tag{32}
\]

We can now compare this with [29] by writing the sine in terms of exponentials

\[
\psi_p(x) = \frac{a}{2i\sqrt{\pi}(\alpha x)^{1/4}} \left[ e^{i\left[ \frac{2}{3}(\alpha x)^{3/2} + \frac{\pi}{4} \right]} - e^{-i\left[ \frac{2}{3}(\alpha x)^{3/2} + \frac{\pi}{4} \right]} \right] \tag{34}
\]

Comparing terms, we have
We now have the constants $B$ and $C$, belonging to the WKB wave function for $x > 0$, in terms of the constant $D$ belonging to the WKB function for $x < 0$, which is what we were after. Note that the WKB functions are still valid only for values of $x$ that maintain a respectable distance from the turning point; all we have done is connected the approximations on either side of the turning point. The final form of the WKB function is, for $x > 0$:

$$\psi(x) \approx -\frac{iD}{\sqrt{|p(x)|}} \left( e^{i\left(\int p\,dx/h + \pi/4\right)} - e^{-i\left(\int p\,dx/h + \pi/4\right)} \right)$$  \hspace{1cm} (41)$$

$$= \frac{2D}{\sqrt{|p(x)|}} \frac{1}{2i} \left( e^{i\left(\int p\,dx/h + \pi/4\right)} - e^{-i\left(\int p\,dx/h + \pi/4\right)} \right)$$  \hspace{1cm} (42)$$

$$= \frac{2D}{\sqrt{|p(x)|}} \sin \left[ \int_0^x p\,dx'/h + \pi/4 \right]$$  \hspace{1cm} (43)$$

And for $x < 0$

$$\psi(x) \approx \frac{D}{\sqrt{|p(x)|}} \exp \left[ -\int_x^0 |p(x')|\,dx'/h \right]$$  \hspace{1cm} (44)$$

If the turning point is at the general location of $x_1$, we can just replace the 0 in the limits of the integrals by $x_1$. The only difference between the cases of increasing and decreasing potentials is that the limits of integration are reversed.

PINGBACKS

Pingback: WKB approximation of the harmonic oscillator
Pingback: WKB approximation for a barrier with sloping sides
Pingback: WKB approximation of a double potential well: turning points