

WKB APPROXIMATION AND THE POWER LAW POTENTIAL

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 8.11.

We've seen that if we apply the WKB approximation to a potential well and require that the WKB wave functions match up in the region between the turning points, we get the condition

$$\int_{x_1}^{x_2} p(x) dx = \left(n - \frac{1}{2}\right) \pi \hbar \quad (1)$$

where x_1 is the left turning point, x_2 is the right turning point and $n = 1, 2, 3, \dots$. We've applied this result to the case of the harmonic oscillator, where the potential is $V(x) = \frac{1}{2}m\omega^2x^2$, but it's interesting to work out the more general case where the potential is any power law:

$$V(x) = \alpha|x|^\nu \quad (2)$$

where α and ν are positive constants. The turning points for a particle with energy E are found by solving $E = V(x)$ so we get

$$x_{1,2} = \pm \left(\frac{E}{\alpha}\right)^{1/\nu} \quad (3)$$

Since the potential is an even function, we can write 1 as

$$\int_{x_1}^{x_2} p(x) dx = 2 \int_0^{(E/\alpha)^{1/\nu}} \sqrt{2m(E - \alpha x^\nu)} dx \quad (4)$$

Maple is unable to handle the integral as it stands, so we can help it by using the substitution

$$u = x^\nu \quad (5)$$

$$du = \nu x^{\nu-1} dx \quad (6)$$

$$= \nu u^{1-1/\nu} dx \quad (7)$$

$$dx = \frac{1}{\nu} u^{1/\nu-1} du \quad (8)$$

The turning point in the u coordinates is E/α , so the integral becomes

$$2 \int_0^{(E/\alpha)^{1/\nu}} \sqrt{2m(E - \alpha x^\nu)} dx = \frac{2\sqrt{2m}}{\nu} \int_0^{E/\alpha} \sqrt{E - \alpha u} u^{1/\nu-1} du \quad (9)$$

Maple can do this integral, with the result

$$\frac{2\sqrt{2m}}{\nu} \int_0^{E/\alpha} \sqrt{E - \alpha u} u^{1/\nu-1} du = \frac{\sqrt{2\pi m}}{\nu} \frac{1}{\alpha^{1/\nu}} \frac{\Gamma(\frac{1}{\nu})}{\Gamma(\frac{1}{\nu} + \frac{3}{2})} E^{\frac{1}{\nu} + \frac{1}{2}} \quad (10)$$

$$= \left(n - \frac{1}{2}\right) \pi \hbar \quad (11)$$

Using the gamma function identity

$$\Gamma(z+1) = z\Gamma(z) \quad (12)$$

with $z = 1/\nu$, we can solve for E and simplify the expression slightly:

$$E_n = \left[\left(n - \frac{1}{2}\right) \pi \hbar \frac{\Gamma(\frac{1}{\nu} + \frac{3}{2})}{\Gamma(\frac{1}{\nu})} \frac{\nu \alpha^{1/\nu}}{\sqrt{2\pi m}} \right]^{\frac{2\nu}{\nu+2}} \quad (13)$$

$$= \left[\left(n - \frac{1}{2}\right) \pi \hbar \frac{\Gamma(\frac{1}{\nu} + \frac{3}{2})}{\Gamma(\frac{1}{\nu} + 1)} \frac{\alpha^{1/\nu}}{\sqrt{2\pi m}} \right]^{\frac{2\nu}{\nu+2}} \quad (14)$$

We also have

$$\left(\alpha^{1/\nu}\right)^{\frac{2\nu}{\nu+2}} = \alpha^{\frac{2}{\nu+2}} = \alpha^{\frac{\nu+2-\nu}{\nu+2}} = \alpha \cdot \alpha^{-\frac{\nu}{\nu+2}} = \alpha \cdot \left[(\alpha)^{-1/2}\right]^{\frac{2\nu}{\nu+2}} \quad (15)$$

so

$$E_n = \alpha \left[\left(n - \frac{1}{2}\right) \hbar \frac{\Gamma(\frac{1}{\nu} + \frac{3}{2})}{\Gamma(\frac{1}{\nu} + 1)} \frac{\sqrt{\pi}}{\sqrt{2m\alpha}} \right]^{\frac{2\nu}{\nu+2}} \quad (16)$$

In the special case of the harmonic oscillator, $\nu = 2$ and $\alpha = \frac{1}{2}m\omega^2$. The values of the gamma function required here are

$$\Gamma(2) = 1 \quad (17)$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi} \quad (18)$$

so we get

$$E_n = \left(n - \frac{1}{2} \right) \hbar \omega \quad (19)$$