WKB APPROXIMATION AND THE POWER LAW POTENTIAL

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 8.11.

We've seen that if we apply the WKB approximation to a potential well and require that the WKB wave functions match up in the region between the turning points, we get the condition

$$\int_{x_1}^{x_2} p(x)\ dx = \left(n - \frac{1}{2}\right)\pi\hbar \tag{0.1}$$

where x_1 is the left turning point, x_2 is the right turning point and $n = 1, 2, 3, \ldots$ We've applied this result to the case of the harmonic oscillator, where the potential is $V(x) = \frac{1}{2}m\omega^2 x^2$, but it's interesting to work out the more general case where the potential is any power law:

$$V(x) = \alpha |x|^{\nu} \tag{0.2}$$

where α and ν are positive constants. The turning points for a particle with energy E are found by solving $E = V(x)$ so we get

$$x_{1,2} = \pm\left(\frac{E}{\alpha}\right)^{1/\nu} \tag{0.3}$$

Since the potential is an even function, we can write 0.1 as

$$\int_{x_1}^{x_2} p(x)\ dx = 2\int_0^{(E/\alpha)^{1/\nu}} \sqrt{2m(E - \alpha x^{\nu})}dx \tag{0.4}$$

Maple is unable to handle the integral as it stands, so we can help it by using the substitution

$$u = x^\nu \tag{0.5}$$
$$du = \nu x^{\nu-1} dx \tag{0.6}$$
$$= \nu u^{1-1/\nu} dx \tag{0.7}$$
$$dx = \frac{1}{\nu} u^{1/\nu-1} du \tag{0.8}$$

The turning point in the u coordinates is E/α, so the integral becomes

$$2\int_0^{(E/\alpha)^{1/\nu}} \sqrt{2m(E-\alpha x^\nu)}dx = \frac{2\sqrt{2m}}{\nu}\int_0^{E/\alpha}\sqrt{E-\alpha u}u^{1/\nu-1}du \tag{0.9}$$

Maple can do this integral, with the result

$$\frac{2\sqrt{2m}}{\nu}\int_0^{E/\alpha}\sqrt{E-\alpha u}u^{1/\nu-1}du = \frac{\sqrt{2\pi m}}{\nu}\frac{1}{\alpha^{1/\nu}}\frac{\Gamma\left(\frac{1}{\nu}\right)}{\Gamma\left(\frac{1}{\nu}+\frac{3}{2}\right)}E^{\frac{1}{\nu}+\frac{1}{2}} \tag{0.10}$$
$$= \left(n-\frac{1}{2}\right)\pi\hbar \tag{0.11}$$

Using the gamma function identity

$$\Gamma(z+1) = z\Gamma(z) \tag{0.12}$$

with $z = 1/\nu$, we can solve for E and simplify the expression slightly:

$$E_n = \left[\left(n-\frac{1}{2}\right)\pi\hbar\frac{\Gamma\left(\frac{1}{\nu}+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{\nu}\right)}\frac{\nu\alpha^{1/\nu}}{\sqrt{2\pi m}}\right]^{\frac{2\nu}{\nu+2}} \tag{0.13}$$
$$= \left[\left(n-\frac{1}{2}\right)\pi\hbar\frac{\Gamma\left(\frac{1}{\nu}+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{\nu}+1\right)}\frac{\alpha^{1/\nu}}{\sqrt{2\pi m}}\right]^{\frac{2\nu}{\nu+2}} \tag{0.14}$$

We also have

$$\left(\alpha^{1/\nu}\right)^{\frac{2\nu}{\nu+2}} = \alpha^{\frac{2}{\nu+2}} = \alpha^{\frac{\nu+2-\nu}{\nu+2}} = \alpha\cdot\alpha^{-\frac{\nu}{\nu+2}} = \alpha\cdot\left[(\alpha)^{-1/2}\right]^{\frac{2\nu}{\nu+2}} \tag{0.15}$$

so

$$E_n = \alpha \left[\left(n - \frac{1}{2} \right) \hbar \frac{\Gamma\left(\frac{1}{\nu} + \frac{3}{2}\right)}{\Gamma\left(\frac{1}{\nu} + 1\right)} \frac{\sqrt{\pi}}{\sqrt{2m\alpha}} \right]^{\frac{2\nu}{\nu+2}} \tag{0.16}$$

In the special case of the harmonic oscillator, $\nu = 2$ and $\alpha = \frac{1}{2}m\omega^2$. The values of the gamma function required here are

$$\Gamma(2) = 1 \tag{0.17}$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi} \tag{0.18}$$

so we get

$$E_n = \left(n - \frac{1}{2} \right) \hbar\omega \tag{0.19}$$