WKB APPROXIMATION AND THE POWER LAW POTENTIAL

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 8.11.

We've seen that if we apply the WKB approximation to a potential well and require that the WKB wave functions match up in the region between the turning points, we get the condition

$$\int_{x_1}^{x_2} p(x)\, dx = \left(n - \frac{1}{2}\right)\pi\hbar \tag{1}$$

where x_1 is the left turning point, x_2 is the right turning point and $n = 1,2,3,\ldots$ We've applied this result to the case of the harmonic oscillator, where the potential is $V(x) = \frac{1}{2}m\omega^2 x^2$, but it's interesting to work out the more general case where the potential is any power law:

$$V(x) = \alpha |x|^{\nu} \tag{2}$$

where α and ν are positive constants. The turning points for a particle with energy E are found by solving $E = V(x)$ so we get

$$x_{1,2} = \pm\left(\frac{E}{\alpha}\right)^{1/\nu} \tag{3}$$

Since the potential is an even function, we can write 1 as

$$\int_{x_1}^{x_2} p(x)\, dx = 2\int_0^{(E/\alpha)^{1/\nu}} \sqrt{2m(E - \alpha x^{\nu})}dx \tag{4}$$

Maple is unable to handle the integral as it stands, so we can help it by using the substitution

$$\begin{aligned} u &= x^{\nu} & (5)\\ du &= \nu x^{\nu-1}dx & (6)\\ &= \nu u^{1-1/\nu}dx & (7)\\ dx &= \frac{1}{\nu}u^{1/\nu-1}du & (8) \end{aligned}$$

The turning point in the u coordinates is E/α, so the integral becomes

$$2\int_0^{(E/\alpha)^{1/\nu}}\sqrt{2m(E-\alpha x^\nu)}dx=\frac{2\sqrt{2m}}{\nu}\int_0^{E/\alpha}\sqrt{E-\alpha u}u^{1/\nu-1}du \tag{9}$$

Maple can do this integral, with the result

$$\frac{2\sqrt{2m}}{\nu}\int_0^{E/\alpha}\sqrt{E-\alpha u}u^{1/\nu-1}du = \frac{\sqrt{2\pi m}}{\nu}\frac{1}{\alpha^{1/\nu}}\frac{\Gamma\left(\frac{1}{\nu}\right)}{\Gamma\left(\frac{1}{\nu}+\frac{3}{2}\right)}E^{\frac{1}{\nu}+\frac{1}{2}} \tag{10}$$

$$= \left(n-\frac{1}{2}\right)\pi\hbar \tag{11}$$

Using the gamma function identity

$$\Gamma(z+1)=z\Gamma(z) \tag{12}$$

with $z=1/\nu$, we can solve for E and simplify the expression slightly:

$$E_n = \left[\left(n-\frac{1}{2}\right)\pi\hbar\frac{\Gamma\left(\frac{1}{\nu}+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{\nu}\right)}\frac{\nu\alpha^{1/\nu}}{\sqrt{2\pi m}}\right]^{\frac{2\nu}{\nu+2}} \tag{13}$$

$$= \left[\left(n-\frac{1}{2}\right)\pi\hbar\frac{\Gamma\left(\frac{1}{\nu}+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{\nu}+1\right)}\frac{\alpha^{1/\nu}}{\sqrt{2\pi m}}\right]^{\frac{2\nu}{\nu+2}} \tag{14}$$

We also have

$$\left(\alpha^{1/\nu}\right)^{\frac{2\nu}{\nu+2}}=\alpha^{\frac{2}{\nu+2}}=\alpha^{\frac{\nu+2-\nu}{\nu+2}}=\alpha\cdot\alpha^{-\frac{\nu}{\nu+2}}=\alpha\cdot\left[(\alpha)^{-1/2}\right]^{\frac{2\nu}{\nu+2}} \tag{15}$$

so

$$E_n=\alpha\left[\left(n-\frac{1}{2}\right)\hbar\frac{\Gamma\left(\frac{1}{\nu}+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{\nu}+1\right)}\frac{\sqrt{\pi}}{\sqrt{2m\alpha}}\right]^{\frac{2\nu}{\nu+2}} \tag{16}$$

In the special case of the harmonic oscillator, $\nu=2$ and $\alpha=\frac{1}{2}m\omega^2$. The values of the gamma function required here are

$$\Gamma(2) = 1 \tag{17}$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi} \tag{18}$$

so we get

$$E_n = \left(n - \frac{1}{2}\right)\hbar\omega \tag{19}$$