

WKB APPROXIMATION AND THE REFLECTIONLESS POTENTIAL

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 8.12.

We've seen that if we apply the WKB approximation to a potential well and require that the WKB wave functions match up in the region between the turning points, we get the condition

$$\int_{x_1}^{x_2} p(x) dx = \left(n - \frac{1}{2}\right) \pi \hbar \quad (1)$$

where x_1 is the left turning point, x_2 is the right turning point and $n = 1, 2, 3, \dots$. Another example of this process is an application to the reflectionless potential that we considered earlier:

$$V(x) = -\frac{\hbar^2 a^2}{m} \operatorname{sech}^2(ax) \quad (2)$$

This gives a potential well centred at $x = 0$ which approaches $V(x) = 0$ as $x \rightarrow \pm\infty$. (It's called 'reflectionless' since if $E > 0$, an incident particle passes straight through the potential without any reflection.)

We'll consider the bound state and compare the WKB approximation to the exact answer, which we worked out as

$$E_0 = -\frac{\hbar^2 a^2}{2m} \quad (3)$$

To apply WKB, we need the turning points x_1 and x_2 where $E = V$. Since V is even, $x_1 = -x_2$ where

$$E = -\frac{\hbar^2 a^2}{m} \operatorname{sech}^2(ax_2) \quad (4)$$

Since V is an even function, we can write 1 as

$$\int_{x_1}^{x_2} p(x) dx = 2 \int_0^{x_2} \sqrt{2m \left(E + \frac{\hbar^2 a^2}{m} \operatorname{sech}^2(ax) \right)} dx \quad (5)$$

Not surprisingly, Maple balks at this integral, so we need to help it along by trying a substitution. We can try

$$z \equiv \operatorname{sech}^2(ax) \quad (6)$$

$$dz = -2a \operatorname{sech}^2(ax) \tanh(ax) dx \quad (7)$$

$$= -2az\sqrt{1-z} dx \quad (8)$$

From 4, we get the limits in terms of z . For $x = 0$, $z = 1$ and for $x = x_2$ we have

$$z_2 = \operatorname{sech}^2(ax_2) \quad (9)$$

$$= -\frac{mE}{\hbar^2 a^2} \equiv b \quad (10)$$

so

$$\int_{x_1}^{x_2} p(x) dx = -2 \int_1^b \sqrt{2m \left(E + \frac{\hbar^2 a^2}{m} z \right)} \frac{dz}{2az\sqrt{1-z}} \quad (11)$$

$$= \sqrt{2\hbar} \int_b^1 \frac{\sqrt{z-b}}{z\sqrt{1-z}} dz \quad (12)$$

Maple can do this integral provided we assume that $b > 0$ (which is true from 10 since $E < 0$) and $b \neq 1$ ($b = 1$ just gives zero for the integral anyway). We get

$$\sqrt{2\hbar} \int_b^1 \frac{\sqrt{z-b}}{z\sqrt{1-z}} dz = \sqrt{2\pi\hbar} (1 - \sqrt{b}) \quad (13)$$

From 1 we have

$$\sqrt{2\pi\hbar} (1 - \sqrt{b}) = \left(n - \frac{1}{2} \right) \pi\hbar \quad (14)$$

$$n = \sqrt{2} (1 - \sqrt{b}) + \frac{1}{2} \quad (15)$$

Since the smallest \sqrt{b} can be is zero, the largest n can be is the greatest integer less than or equal to $\sqrt{2} + \frac{1}{2} = 1.914$, so the only possible value of n is $n = 1$. In that case

$$\sqrt{2}(1 - \sqrt{b}) = \frac{1}{2} \quad (16)$$

$$b = \left(1 - \frac{1}{2\sqrt{2}}\right)^2 \quad (17)$$

$$= \left(\frac{9}{8} - \frac{1}{\sqrt{2}}\right) \quad (18)$$

$$E = -\frac{\hbar^2 a^2}{m} \left(\frac{9}{8} - \frac{1}{\sqrt{2}}\right) \quad (19)$$

$$\approx -0.418 \frac{\hbar^2 a^2}{m} \quad (20)$$

Comparing this to 3, we see that WKB gives a reasonable approximation.