# WKB APPROXIMATION AND THE REFLECTIONLESS POTENTIAL

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 8.12.

We've seen that if we apply the WKB approximation to a potential well and require that the WKB wave functions match up in the region between the turning points, we get the condition

$$\int_{x_1}^{x_2} p(x)\,dx = \left(n-\frac{1}{2}\right)\pi\hbar \tag{1}$$

where $x_1$ is the left turning point, $x_2$ is the right turning point and $n = 1,2,3,\ldots$ Another example of this process is an application to the reflectionless potential that we considered earlier:

$$V(x) = -\frac{\hbar^2 a^2}{m}\operatorname{sech}^2(ax) \tag{2}$$

This gives a potential well centred at $x=0$ which approaches $V(x)=0$ as $x\to\pm\infty$. (It's called 'reflectionless' since if $E>0$, an incident particle passes straight through the potential without any reflection.)

We'll consider the bound state and compare the WKB approximation to the exact answer, which we worked out as

$$E_0 = -\frac{\hbar^2 a^2}{2m} \tag{3}$$

To apply WKB, we need the turning points $x_1$ and $x_2$ where $E=V$. Since $V$ is even, $x_1=-x_2$ where

$$E = -\frac{\hbar^2 a^2}{m}\operatorname{sech}^2(ax_2) \tag{4}$$

Since $V$ is an even function, we can write 1 as

$$\int_{x_1}^{x_2} p(x)\,dx = 2\int_0^{x_2}\sqrt{2m\left(E+\frac{\hbar^2 a^2}{m}\operatorname{sech}^2(ax)\right)}dx \tag{5}$$

Not surprisingly, Maple balks at this integral, so we need to help it along by trying a substitution. We can try

$$\begin{aligned} z &\equiv \operatorname{sech}^2(ax) && (6)\\ dz &= -2a\operatorname{sech}^2(ax)\tanh(ax)\,dx && (7)\\ &= -2az\sqrt{1-z}dx && (8) \end{aligned}$$

From 4, we get the limits in terms of $z$. For $x=0$, $z=1$ and for $x=x_2$ we have

$$\begin{aligned} z_2 &= \operatorname{sech}^2(ax_2) && (9)\\ &= -\frac{mE}{\hbar^2a^2}\equiv b && (10) \end{aligned}$$

so

$$\begin{aligned} \int_{x_1}^{x_2} p(x)\;dx &= -2\int_1^b \sqrt{2m\left(E+\frac{\hbar^2a^2}{m}z\right)}\frac{dz}{2az\sqrt{1-z}} && (11)\\ &= \sqrt{2}\hbar\int_b^1 \frac{\sqrt{z-b}}{z\sqrt{1-z}}dz && (12) \end{aligned}$$

Maple can do this integral provided we assume that $b>0$ (which is true from 10 since $E<0$) and $b\neq 1$ ($b=1$ just gives zero for the integral anyway). We get

$$\sqrt{2}\hbar\int_b^1 \frac{\sqrt{z-b}}{z\sqrt{1-z}}dz = \sqrt{2}\pi\hbar\left(1-\sqrt{b}\right) \tag{13}$$

From 1 we have

$$\begin{aligned} \sqrt{2}\pi\hbar\left(1-\sqrt{b}\right) &= \left(n-\frac{1}{2}\right)\pi\hbar && (14)\\ n &= \sqrt{2}\left(1-\sqrt{b}\right)+\frac{1}{2} && (15) \end{aligned}$$

Since the smallest $\sqrt{b}$ can be is zero, the largest $n$ can be is the greatest integer less than or equal to $\sqrt{2}+\frac{1}{2}=1.914$, so the only possible value of $n$ is $n=1$. In that case

$$\sqrt{2}\left(1-\sqrt{b}\right) = \frac{1}{2} \tag{16}$$

$$b = \left(1-\frac{1}{2\sqrt{2}}\right)^2 \tag{17}$$

$$= \left(\frac{9}{8}-\frac{1}{\sqrt{2}}\right) \tag{18}$$

$$E = -\frac{\hbar^2 a^2}{m}\left(\frac{9}{8}-\frac{1}{\sqrt{2}}\right) \tag{19}$$

$$\approx -0.418\frac{\hbar^2 a^2}{m} \tag{20}$$

Comparing this to 3, we see that WKB gives a reasonable approximation.