

WKB APPROXIMATION AND THE REFLECTIONLESS POTENTIAL

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 8.12.

We've seen that if we apply the WKB approximation to a potential well and require that the WKB wave functions match up in the region between the turning points, we get the condition

$$(0.1) \quad \int_{x_1}^{x_2} p(x) dx = \left(n - \frac{1}{2}\right) \pi \hbar$$

where x_1 is the left turning point, x_2 is the right turning point and $n = 1, 2, 3, \dots$. Another example of this process is an application to the reflectionless potential that we considered earlier:

$$(0.2) \quad V(x) = -\frac{\hbar^2 a^2}{m} \operatorname{sech}^2(ax)$$

This gives a potential well centred at $x = 0$ which approaches $V(x) = 0$ as $x \rightarrow \pm\infty$. (It's called 'reflectionless' since if $E > 0$, an incident particle passes straight through the potential without any reflection.)

We'll consider the bound state and compare the WKB approximation to the exact answer, which we worked out as

$$(0.3) \quad E_0 = -\frac{\hbar^2 a^2}{2m}$$

To apply WKB, we need the turning points x_1 and x_2 where $E = V$. Since V is even, $x_1 = -x_2$ where

$$(0.4) \quad E = -\frac{\hbar^2 a^2}{m} \operatorname{sech}^2(ax_2)$$

Since V is an even function, we can write 0.1 as

$$(0.5) \quad \int_{x_1}^{x_2} p(x) dx = 2 \int_0^{x_2} \sqrt{2m \left(E + \frac{\hbar^2 a^2}{m} \operatorname{sech}^2(ax) \right)} dx$$

Not surprisingly, Maple balks at this integral, so we need to help it along by trying a substitution. We can try

$$(0.6) \quad z \equiv \operatorname{sech}^2(ax)$$

$$(0.7) \quad dz = -2a \operatorname{sech}^2(ax) \tanh(ax) dx$$

$$(0.8) \quad = -2az\sqrt{1-z} dx$$

From 0.4, we get the limits in terms of z . For $x = 0$, $z = 1$ and for $x = x_2$ we have

$$(0.9) \quad z_2 = \operatorname{sech}^2(ax_2)$$

$$(0.10) \quad = -\frac{mE}{\hbar^2 a^2} \equiv b$$

so

$$(0.11) \quad \int_{x_1}^{x_2} p(x) dx = -2 \int_1^b \sqrt{2m \left(E + \frac{\hbar^2 a^2}{m} z \right)} \frac{dz}{2az\sqrt{1-z}}$$

$$(0.12) \quad = \sqrt{2\hbar} \int_b^1 \frac{\sqrt{z-b}}{z\sqrt{1-z}} dz$$

Maple can do this integral provided we assume that $b > 0$ (which is true from 0.10 since $E < 0$) and $b \neq 1$ ($b = 1$ just gives zero for the integral anyway). We get

$$(0.13) \quad \sqrt{2\hbar} \int_b^1 \frac{\sqrt{z-b}}{z\sqrt{1-z}} dz = \sqrt{2\pi\hbar} (1 - \sqrt{b})$$

From 0.1 we have

$$(0.14) \quad \sqrt{2\pi\hbar} (1 - \sqrt{b}) = \left(n - \frac{1}{2} \right) \pi\hbar$$

$$(0.15) \quad n = \sqrt{2} (1 - \sqrt{b}) + \frac{1}{2}$$

Since the smallest \sqrt{b} can be is zero, the largest n can be is the greatest integer less than or equal to $\sqrt{2} + \frac{1}{2} = 1.914$, so the only possible value of n is $n = 1$. In that case

$$(0.16) \quad \sqrt{2}(1 - \sqrt{b}) = \frac{1}{2}$$

$$(0.17) \quad b = \left(1 - \frac{1}{2\sqrt{2}}\right)^2$$

$$(0.18) \quad = \left(\frac{9}{8} - \frac{1}{\sqrt{2}}\right)$$

$$(0.19) \quad E = -\frac{\hbar^2 a^2}{m} \left(\frac{9}{8} - \frac{1}{\sqrt{2}}\right)$$

$$(0.20) \quad \approx -0.418 \frac{\hbar^2 a^2}{m}$$

Comparing this to 0.3, we see that WKB gives a reasonable approximation.