WKB APPROXIMATION AND THE RADIAL EQUATION

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So far, we’ve applied the WKB approximation to one-dimensional potential problems. It might seem that that’s all we can manage since WKB is essentially a way of approximating the solution of a one-dimensional ODE. However, we can use it on 3-d problems in those cases where the solution is separable, such as spherically symmetric potentials. For such potentials, the general wave function can be written as the product of a radial function $R(r)$ and a spherical harmonic $Y(\theta, \phi)$. With the substitution $u(r) \equiv rR(r)$ we found that the radial equation can be written as

$$ -\frac{\hbar^2}{2m} \frac{d^2u}{dr^2} + \left( V(r) + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} \right) u = Eu $$

(1)

For the simplest case, we take $l = 0$ so the equation becomes

$$ -\frac{\hbar^2}{2m} \frac{d^2u}{dr^2} + V(r) u = Eu $$

(2)

which has exactly the same form as the one-dimensional Schrödinger equation, so we should be able to use WKB to get approximate solutions.

There is one detail we need to work out first, though. In applying WKB to a true 1-d situation, the $x$ coordinate extended to infinity in both directions which allowed us to discard exponential terms that blow up as we approach one extreme or the other. In this equation, the $r$ coordinate starts at 0. One way of handling this is to assume that there is an infinite wall at $r = 0$ so that $u(0) = 0$. Since $u(r) = rR(r)$, this is a reasonable assumption, since it requires only that $R(0)$ is finite.

We can therefore consider a spherically symmetric potential well with an infinite barrier at $r = 0$ and then some potential that increases from $r = 0$ out to $r = \infty$. For a given energy $E$, there will be one turning point $r_2$ where $E = V(r_2)$.

For a turning point where $V$ is increasing, we’ve seen that the WKB functions on either side of the turning point are
\[ u(r) = \begin{cases} \frac{2D}{\sqrt{p(r)}} \sin \left[ \int_{r}^{r_{2}} p(r') \, dr'/\hbar + \pi/4 \right] & r < r_{2} \\ \frac{D}{\sqrt{|p(r)|}} \exp \left[ -\int_{r_{2}}^{r} |p(r')| \, dr'/\hbar \right] & r > r_{2} \end{cases} \] (3)

The requirement \( u(0) = 0 \) means that the sine must be zero at \( r = 0 \), so

\[ \int_{0}^{r_{2}} p(r) \, dr / \hbar + \frac{\pi}{4} = n\pi \] (4)

\[ \int_{0}^{r_{2}} p(r) \, dr = \left( n - \frac{1}{4} \right) \pi \hbar \] (5)

**Example.** We can apply this formula to the potential

\[ V(r) = V_{0} \ln \frac{r}{a} \] (6)

The turning point is defined by

\[ E = V_{0} \ln \frac{r_{2}}{a} \] (7)

so the integral 5 is

\[ \sqrt{2m} \int_{0}^{r_{2}} \sqrt{E - V_{0} \ln \frac{r}{a}} \, dr = \sqrt{2m} \int_{0}^{r_{2}} \sqrt{V_{0} \ln \frac{r_{2}}{a} - V_{0} \ln \frac{r}{a}} \, dr \] (8)

\[ = \sqrt{2mV_{0}} \int_{0}^{r_{2}} \sqrt{\ln \frac{r_{2}}{r}} \, dr \] (9)

We can use the substitution

\[ v = \ln \frac{r_{2}}{r} \] (10)

\[ dv = \frac{r}{r_{0}} \left( -\frac{r_{0}}{r_{2}} \right) \, dr \] (11)

\[ = -\frac{1}{r} \, dr \] (12)

\[ = -\frac{e^{v}}{r_{0}} \, dr \] (13)

The limits on the integral in terms of \( v \) are

\[ r = 0 \rightarrow u = \infty \] (14)

\[ r = r_{2} \rightarrow u = 0 \] (15)
so the integral transforms as

\[
\sqrt{2mV_0} \int_0^{r_2} \sqrt{\ln \frac{r_2}{r}} dr = r_2 \sqrt{2mV_0} \int_0^{\infty} \sqrt{ve^{-v}} dv
\]

(16)

\[
= r_2 \sqrt{2mV_0} \Gamma \left( \frac{3}{2} \right)
\]

(17)

\[
= \frac{\sqrt{2\pi mV_0 r_2}}{2}
\]

(18)

\[
= \left( n - \frac{1}{4} \right) \pi \hbar
\]

(19)

where we used 5 in the last line.

To get the allowed energies we can substitute for \( r_2 \) using 7:

\[
r_2 = ae^{E/V_0}
\]

(20)

\[
= \sqrt{\frac{2\pi}{mV_0}} \left( n - \frac{1}{4} \right) \frac{\hbar}{a}
\]

(21)

\[
E_n = V_0 \ln \left( \sqrt{\frac{2\pi}{mV_0}} \left( n - \frac{1}{4} \right) \frac{\hbar}{a} \right)
\]

(22)

The spacing between successive energy levels is

\[
E_{n+1} - E_n = V_0 \left[ \ln \left( \sqrt{\frac{2\pi}{mV_0}} \left( n + \frac{3}{4} \right) \frac{\hbar}{a} \right) - \ln \left( \sqrt{\frac{2\pi}{mV_0}} \left( n - \frac{1}{4} \right) \frac{\hbar}{a} \right) \right]
\]

(23)

\[
= V_0 \ln \left( \frac{n + 3/4}{n - 1/4} \right)
\]

(24)

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