

STARK EFFECT: TUNNELLING PROBABILITY

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 8.16.

Earlier, we analyzed the Stark effect in hydrogen using perturbation theory. The Stark effect causes a splitting of the spectral lines of hydrogen when an external electric field is applied. In our earlier post, we did a 'proper' analysis by using the correct Coulomb potential for the interaction between the proton and electron, but we can use a cruder model in which we treat the Coulomb attraction as a deep, but finite, square well with the bottom at zero and the top at an energy of V_0 . If the depth V_0 of the well satisfies $V_0 \gg \hbar^2/ma^2$ (where $2a$ is the width of the well) then the bound state energy levels are given by

$$E \approx \frac{n^2\pi^2\hbar^2}{2m(2a)^2} \quad (1)$$

The ground state energy is thus

$$E_1 \approx \frac{\pi^2\hbar^2}{8ma^2} \quad (2)$$

Now suppose we add a weak electric field $\mathbf{E} = -E_{ext}\hat{\mathbf{x}}$, so the electron feels a force $\mathbf{F} = eE_{ext}\hat{\mathbf{x}}$ (in the $+x$ direction). To move the electron at constant speed from x_1 to $x_2 > x_1$ we must apply a force $-eE_{ext}\hat{\mathbf{x}}$ (to prevent the electron from accelerating due to the electric field) over the interval so the work we do is

$$W = \int_{x_1}^{x_2} \mathbf{F} \cdot d\mathbf{x} \quad (3)$$

$$= -eE_{ext}(x_2 - x_1) \quad (4)$$

The potential energy due to E_{ext} is, if we take it to be zero at $x = 0$

$$V_{ext} = -eE_{ext}x \quad (5)$$

Adding in this potential as a perturbation, the net potential for the electron is

$$V(x) = \begin{cases} -eE_{ext}x & -a < x < a \\ V_0 - eE_{ext}x & x > a \end{cases} \quad (6)$$

The shape of this potential is a square well with a bottom that slopes downwards from left to right with a slope of $-eE_{ext}$, and a top that slopes downwards from a height of V_0 from $x = a$ onwards. Thus for any bound state energy E_n , there is a point $x_t = (V_0 - E_n)/eE_{ext}$ where $E_n = V(x_t)$ and the particle could, in principle, tunnel out of the well and escape in the $+x$ direction. (An escape in the $-x$ direction isn't possible since $V(x)$ increases from V_0 as x goes to the left of $x = -a$.)

We can use the WKB approximation for a particle tunneling through a barrier to see how likely this is to occur. The formula for the transmission probability T is

$$T \approx e^{-2\gamma} \quad (7)$$

where

$$\gamma = \frac{1}{\hbar} \int_{x_1}^{x_2} |p(x)| dx \quad (8)$$

and the barrier extends from $x = x_1$ to $x = x_2$. In this case the barrier extends from $x_1 = a$ to $x_2 = x_t = (V_0 - E_n)/eE_{ext}$ so we have

$$\gamma = \frac{1}{\hbar} \int_a^{(V_0 - E_n)/eE_{ext}} \sqrt{2m(V_0 - eE_{ext}x - E_n)} dx \quad (9)$$

$$= \frac{2\sqrt{2m}}{3\hbar eE_{ext}} (V_0 - eE_{ext}a - E_n)^{3/2} \quad (10)$$

$$\approx \frac{2\sqrt{2m}}{3\hbar eE_{ext}} V_0^{3/2} \quad (11)$$

where the last line assumes $V_0 \gg eE_{ext}a + E_n$.

In our study of alpha decay we got an estimate of the half-life of a particle as (equation 34 there)

$$\tau \approx \frac{2r_1}{v} e^{2\gamma} \quad (12)$$

where r_1 is the distance the electron must travel to reach the tipping point and v is the speed of the electron. To get the speed of an electron in this potential we can take particle's kinetic energy to be equal to its total energy, so

$$\frac{1}{2}mv^2 = E_1 \quad (13)$$

$$v = \frac{\pi\hbar}{2meE_{ext}} \quad (14)$$

Then

$$\tau = \frac{2r_1}{v}e^{2\gamma} \quad (15)$$

$$= \frac{8m(eE_{ext})^2}{\pi\hbar}e^{2\gamma} \quad (16)$$

Plugging in the values given by Griffiths in the question:

$$V_0 = 20 \text{ eV} \quad (17)$$

$$a = 10^{-10} \text{ m} \quad (18)$$

$$E_{ext} = 7 \times 10^6 \text{ V m}^{-1} \quad (19)$$

$$e = 1.602 \times 10^{-19} \text{ C} \quad (20)$$

$$m = 9.11 \times 10^{-31} \text{ kg} \quad (21)$$

we get

$$\gamma = 43641 \quad (22)$$

$$\tau = 2.47 \times 10^{37890} \text{ sec} \quad (23)$$

As the age of the universe is around 10^{17} sec, this isn't a tunneling event we can expect to see any time soon.