STARK EFFECT: TUNNELLING PROBABILITY

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 8.16.

Earlier, we analyzed the Stark effect in hydrogen using perturbation theory. The Stark effect causes a splitting of the spectral lines of hydrogen when an external electric field is applied. In our earlier post, we did a 'proper' analysis by using the correct Coulomb potential for the interaction between the proton and electron, but we can use a cruder model in which we treat the Coulomb attraction as a deep, but finite, square well with the bottom at zero and the top at an energy of V_0. If the depth V_0 of the well satisfies $V_0 \gg \hbar^2/ma^2$ (where $2a$ is the width of the well) then the bound state energy levels are given by

$$E \approx \frac{n^2\pi^2\hbar^2}{2m(2a)^2} \tag{1}$$

The ground state energy is thus

$$E_1 \approx \frac{\pi^2\hbar^2}{8ma^2} \tag{2}$$

Now suppose we add a weak electric field $\mathbf{E} = -E_{ext}\hat{\mathbf{x}}$, so the electron feels a force $\mathbf{F} = eE_{ext}\hat{\mathbf{x}}$ (in the $+x$ direction). To move the electron at constant speed from x_1 to $x_2 > x_1$ we must apply a force $-eE_{ext}\hat{\mathbf{x}}$ (to prevent the electron from accelerating due to the electric field) over the interval so the work we do is

$$\begin{aligned} W &= \int_{x_1}^{x_2} \mathbf{F}\cdot d\mathbf{x} & (3)\\ &= -eE_{ext}(x_2 - x_1) & (4) \end{aligned}$$

The potential energy due to E_{ext} is, if we take it to be zero at $x = 0$

$$V_{ext} = -eE_{ext}x \tag{5}$$

Adding in this potential as a perturbation, the net potential for the electron is

$$V(x)=\begin{cases}-eE_{ext}x & -a<x<a\\ V_0-eE_{ext}x & x>a\end{cases} \tag{6}$$

The shape of this potential is a square well with a bottom that slopes downwards from left to right with a slope of $-eE_{ext}$, and a top that slopes downwards from a height of V_0 from $x=a$ onwards. Thus for any bound state energy E_n, there is a point $x_t=(V_0-E_n)/eE_{ext}$ where $E_n=V(x_t)$ and the particle could, in principle, tunnel out of the well and escape in the $+x$ direction. (An escape in the $-x$ direction isn't possible since $V(x)$ increases from V_0 as x goes to the left of $x=-a$.)

We can use the WKB approximation for a particle tunneling through a barrier to see how likely this is to occur. The formula for the transmission probability T is

$$T\approx e^{-2\gamma} \tag{7}$$

where

$$\gamma=\frac{1}{\hbar}\int_{x_1}^{x_2}|p(x)|\,dx \tag{8}$$

and the barrier extends from $x=x_1$ to $x=x_2$. In this case the barrier extends from $x_1=a$ to $x_2=x_t=(V_0-E_n)/eE_{ext}$ so we have

$$\begin{aligned}\gamma &= \frac{1}{\hbar}\int_{a}^{(V_0-E_n)/eE_{ext}}\sqrt{2m(V_0-eE_{ext}x-E_n)}dx && (9)\\ &= \frac{2\sqrt{2m}}{3\hbar eE_{ext}}(V_0-eE_{ext}a-E_n)^{3/2} && (10)\\ &\approx \frac{2\sqrt{2m}}{3\hbar eE_{ext}}V_0^{3/2} && (11)\end{aligned}$$

where the last line assumes $V_0\gg eE_{ext}a+E_n$.

In our study of alpha decay we got an estimate of the half-life of a particle as (equation 34 there)

$$\tau\approx\frac{2r_1}{v}e^{2\gamma} \tag{12}$$

where r_1 is the distance the electron must travel to reach the tipping point and v is the speed of the electron. To get the speed of an electron in this potential we can take particle's kinetic energy to be equal to its total energy, so

$$\frac{1}{2}mv^2 = E_1 \tag{13}$$

$$v = \frac{\pi\hbar}{2meE_{ext}} \tag{14}$$

Then

$$\tau = \frac{2r_1}{v}e^{2\gamma} \tag{15}$$

$$= \frac{8m\left(eE_{ext}\right)^2}{\pi\hbar}e^{2\gamma} \tag{16}$$

Plugging in the values given by Griffiths in the question:

$$V_0 = 20 \text{ eV} \tag{17}$$

$$a = 10^{-10} \text{ m} \tag{18}$$

$$E_{ext} = 7\times10^6 \text{ V m}^{-1} \tag{19}$$

$$e = 1.602\times10^{-19} \text{ C} \tag{20}$$

$$m = 9.11\times10^{-31} \text{ kg} \tag{21}$$

we get

$$\gamma = 43641 \tag{22}$$

$$\tau = 2.47\times10^{37890} \text{ sec} \tag{23}$$

As the age of the universe is around 10^{17} sec, this isn't a tunneling event we can expect to see any time soon.