HALF LIFE OF A BEER CAN

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 8.17.

Whe we analyzed the WKB approximation for a particle tunneling through a barrier we came up with a formula for the transmission probability T:

$$T \approx e^{-2\gamma} \tag{1}$$

where

$$\gamma = \frac{1}{\hbar}\int_{-a}^{a}|p(x)|\,dx \tag{2}$$

and the barrier extends from $x=-a$ to $x=+a$. We can apply this formula to get an idea of why quantum tunneling isn't a common phenomenon in the macroscopic world.

Suppose we have a can of beer with a mass of $m=0.5$ kg, a diameter of $D=0.06$ m and a height of $h=0.16$ m. If x is the height of the centre of mass of the can above its height when the can is standing upright (that is, at $x=h/2$), then the potential energy of the can is

$$V(x) = mgx \tag{3}$$

If the total energy of the can is $E=0$ then

$$|p(x)| = \sqrt{2m(mgx)} = m\sqrt{2gx} \tag{4}$$

The can is stable in two orientations: on its end (the way it normally sits on a table) and on its side. There is a potential barrier between these two states. If the can starts in an upright position and is tilted, it will spontaneously fall over onto its side when its centre of mass is no longer within the diameter of the can. That is, with the centre of mass at the centre of the can, this happens when

$$x_0 = \sqrt{(D/2)^2 + (h/2)^2} - h/2 \tag{5}$$

The barrier through which we wish to tunnel thus extends from $x=0$ (can standing upright) to $x=x_0$ (can about to tip over). We get

$$\begin{aligned}
\gamma &= \frac{m\sqrt{2g}}{\hbar}\int_0^{x_0}\sqrt{x}dx && (6)\\
&= \frac{2m\sqrt{2g}}{3\hbar}x_0^{3/2} && (7)\\
&= \frac{m\sqrt{g}}{3\hbar}\left(\sqrt{D^2+h^2}-h\right)^{3/2} && (8)
\end{aligned}$$

With the values above we get

$$\gamma = 5.64 \times 10^{30} \tag{9}$$

How long is the 'half-life' of a beer can? In our study of alpha decay we got an estimate of the half-life of a particle as (equation 34 there)

$$\tau \approx \frac{2r_1}{v}e^{2\gamma} \tag{10}$$

where r_1 is the distance the particle (or beer can, in this case) must travel to reach the tipping point and v is the speed of the particle (can). Treating the can as a 'particle', we can its average speed due to thermal motion at room temperature (293 K) as

$$v = \sqrt{\frac{k_BT}{m}} = 8.992 \times 10^{-11} \text{ m s}^{-1} \tag{11}$$

The critical distance is

$$r_1 = x_0 = 0.00544 \text{ m} \tag{12}$$

(Recall this is the upward distance the centre of mass must move to reach the tipping point; it's not the horizontal distance the centre of mass moves. That's why it's only about 5 mm.)

Plugging in the numbers, we get

$$\tau \approx 3.306 \times 10^{-8} e^{10^{31}} \text{ s} \tag{13}$$

That's a very big number. We can get it in powers of 10 using base-10 logarithms:

$$\log_{10}\tau \approx 10^{31}\log_{10}e = 5 \times 10^{30} \tag{14}$$

So the half-life is

$$\tau \approx 10^{5\times 10^{30}} \text{ s} \tag{15}$$

Converting to years won't make much difference, since it will subtract only around 7 from the exponent. The age of the universe is currently estimated to be around 10^{10} years, or around 10^{17} seconds. We can see from this just how unlikely quantum events are in the macroscopic world.