TIME-DEPENDENT SCHRÖDINGER EQUATION: SWITCHING A PERTURBATION ON AND OFF

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 9.2.

We've seen that we can solve the Schrödinger equation with a time-dependent potential in a two-state system if we split the hamiltonian into a time-independent part H^0 and a time-dependent part H', so that the complete hamiltonian is

$$H = H^0 + H' \tag{1}$$

The solution is

$$\mathbf{\Psi}(x,t) = c_a(t)\,\psi_a(x)\,e^{-iE_at/\hbar} + c_b(t)\,\psi_b(x)\,e^{-iE_bt/\hbar} \tag{2}$$

where ψ_a and ψ_b are the two eigenstates of H^0 and the coefficients are solutions of the coupled ODEs

$$\dot{c}_a = -\frac{i}{\hbar}\left[c_a H'_{aa} + c_b H'_{ab} e^{-i(E_b-E_a)t/\hbar}\right] \tag{3}$$

$$\dot{c}_b = -\frac{i}{\hbar}\left[c_b H'_{bb} + c_a H'_{ba} e^{i(E_b-E_a)t/\hbar}\right] \tag{4}$$

where

$$H'_{ij} \equiv \langle\psi_i|\,H'\,|\psi_j\rangle \tag{5}$$

In many problems the diagonal matrix elements are zero, in which case we get

$$\dot{c}_a = -\frac{i}{\hbar} c_b H'_{ab} e^{-i(E_b-E_a)t/\hbar} \equiv -\frac{i}{\hbar} c_b H'_{ab} e^{-i\omega_0 t} \tag{6}$$

$$\dot{c}_b = -\frac{i}{\hbar} c_a H'_{ba} e^{i(E_b-E_a)t/\hbar} \equiv -\frac{i}{\hbar} c_a H'_{ba} e^{i\omega_0 t} \tag{7}$$

In general, we can't get an exact solution of these equations since usually H' is such that the equations can't be integrated in closed form. However, one case that can be solved is that where H' itself doesn't depend on time.

(Actually, it seems rather silly to solve this problem as a time-dependent problem, since if H' doesn't depend on time and is small enough to be considered as a perturbation, we can just use time-independent perturbation theory. In the question, Griffiths explains that what we're really doing in this problem is starting in a system where the hamiltonian is purely H^0 then at time $t=0$ we switch on H', which then remains constant until some later time when we switch it off again, returning to just H^0. During the time that H' is switched on, ψ_a and ψ_b are no longer eigenstates of the full hamiltonian, so the actual wave function is some linear combination of them as given by 2. Thus it's not really a true time-independent problem.)

We can find c_a and c_b by taking the derivative of 6 and then using 7 to eliminate $\dot{c}_b$:

$$\ddot{c}_a = -\frac{i}{\hbar}\dot{c}_b H'_{ab} e^{-i\omega_0 t} - i\omega_0\left(-\frac{i}{\hbar}c_b H'_{ab} e^{-i\omega_0 t}\right) \tag{8}$$

$$= -\frac{i}{\hbar}\left(-\frac{i}{\hbar}c_a H'_{ba} e^{i\omega_0 t}\right) H'_{ab} e^{-i\omega_0 t} - i\omega_0 \dot{c}_a \tag{9}$$

$$\ddot{c}_a + i\omega_0 \dot{c}_a + \frac{\left|H'_{ab}\right|^2}{\hbar^2} c_a = 0 \tag{10}$$

This is now a second order ODE with constant coefficients, so the general solution is

$$c_a(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \tag{11}$$

where $\lambda_{1,2}$ are the roots of the characteristic equation

$$\lambda^2 + i\omega_0\lambda + \frac{\left|H'_{ab}\right|^2}{\hbar^2} = 0 \tag{12}$$

so we have

$$\lambda_{1,2} = -\frac{1}{2}i\omega_0 \pm \frac{1}{2}\sqrt{-\omega_0^2 - \frac{4\left|H'_{ab}\right|^2}{\hbar^2}} \tag{13}$$

$$= -\frac{1}{2}i\omega_0 \pm \frac{i}{2}\sqrt{\omega_0^2 + \frac{4\left|H'_{ab}\right|^2}{\hbar^2}} \tag{14}$$

$$\equiv -\frac{1}{2}i\omega_0 \pm iQ \tag{15}$$

The solution is then

$$c_a(t) = e^{-i\omega_0 t/2}\left(Ae^{iQt} + Be^{-iQt}\right) \tag{16}$$

We can get c_b from 6:

$$\begin{aligned} c_b(t) &= e^{i\omega_0 t}\frac{i\hbar}{H'_{ab}}\dot{c}_a && (17)\\ &= \frac{\hbar}{2H'_{ab}}e^{i\omega_0 t/2}\left[A(-2Q+\omega_0)e^{iQt} + B(2Q+\omega_0)e^{-iQt}\right] && (18)\end{aligned}$$

This is the general solution, but to apply it to a specific case we need to specify H' and the initial conditions. We'll keep H' general, but consider the case where the system starts out in state ψ_a at $t=0$, so that $c_a(0)=1$ and $c_b(0)=0$. Then from 16 and 18 we have

$$\begin{aligned} A+B &= 1 && (19)\\ A(-2Q+\omega_0) &= -B(2Q+\omega_0) && (20)\end{aligned}$$

We can solve these two equations to get

$$\begin{aligned} A &= \frac{2Q+\omega_0}{4Q} && (21)\\ B &= \frac{2Q-\omega_0}{4Q} && (22)\end{aligned}$$

Therefore

$$\begin{aligned} c_a(t) &= \frac{1}{4Q}e^{-i\omega_0 t/2}\left[(2Q+\omega_0)e^{iQt} + (2Q-\omega_0)e^{-iQt}\right] && (23)\\ &= e^{-i\omega_0 t/2}\left[\cos(Qt) + \frac{i\omega_0}{2Q}\sin(Qt)\right] && (24)\\ c_b(t) &= \frac{\hbar}{8H'_{ab}Q}e^{i\omega_0 t/2}\left[\left(\omega_0^2-4Q^2\right)\left(e^{iQt}-e^{-iQt}\right)\right] && (25)\\ &= \frac{i\hbar}{4H'_{ab}Q}e^{i\omega_0 t/2}\left(\omega_0^2-4Q^2\right)\sin(Qt) && (26)\\ &= \frac{i\hbar}{4H'_{ab}Q}e^{i\omega_0 t/2}\left(-\frac{4\left|H'_{ab}\right|^2}{\hbar^2}\right)\sin(Qt) && (27)\\ &= -\frac{i\left|H'_{ab}\right|}{\hbar Q}e^{i\omega_0 t/2}\sin(Qt) && (28)\end{aligned}$$

using 15 in the last two lines.

As a check we can calculate $|c_a|^2+|c_b|^2$.

$$|c_a|^2 = \cos^2(Qt)+\frac{\omega_0^2}{4Q^2}\sin^2(Qt) \tag{29}$$

$$|c_b|^2 = \frac{\left|H'_{ab}\right|^2}{\hbar^2Q^2}\sin^2(Qt) \tag{30}$$

$$|c_a|^2+|c_b|^2 = \cos^2(Qt)+\frac{1}{4Q^2}\left(\omega_0^2+\frac{4\left|H'_{ab}\right|^2}{\hbar^2}\right)\sin^2(Qt) \tag{31}$$

$$= \cos^2(Qt)+\sin^2(Qt) \tag{32}$$

$$= 1 \tag{33}$$

The system oscillates between being entirely in state ψ_a and a mixture of ψ_a and ψ_b with period $2\pi/Q$.

If the perturbation $H'=0$ then $Q=\omega_0/2$ and $H'_{ab}=H'_{ba}=0$ so $c_a=1$ and $c_b=0$ for all times.

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