DELTA FUNCTION IN TIME PERTURBATION

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 9.3.

We've seen that we can solve the Schrödinger equation with a time-dependent potential in a two-state system if we split the hamiltonian into a time-independent part H^0 and a time-dependent part H', so that the complete hamiltonian is

$$H = H^0 + H' \tag{1}$$

The solution is

$$\Psi(x,t) = c_a(t)\,\psi_a(x)\,e^{-iE_a t/\hbar} + c_b(t)\,\psi_b(x)\,e^{-iE_b t/\hbar} \tag{2}$$

As an application of the situation where a perturbation to the hamiltonian in a two-state system is switched on at $t = 0$ and then off again at some later time $t = \tau$ we can look at a perturbation that is on only momentarily, which we can model by using a delta function:

$$H' = U(x)\,\delta(t) \tag{3}$$

We can think of this as the limit as $\tau \to 0$ of the earlier problem if we start with

$$H' = \begin{cases} \frac{U}{\tau} & 0 \le t \le \tau \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

This works because the integral

$$\int_0^\tau \frac{dt}{\tau} = 1 \tag{5}$$

so as $\tau \to 0$ 4 becomes 3.

If we assume that the matrix elements of H' in the two-state system satisfy

$$U_{aa} = U_{bb} = 0 \tag{6}$$

$$U_{ab} = U_{ba}^* \equiv \alpha \tag{7}$$

then in the interval $0 \le t \le \tau$ we have

$$\begin{aligned} H'_{aa} &= H'_{bb} = 0 && (8)\\ H'_{ab} &= \left(H'_{ba}\right)^* = \frac{\alpha}{\tau} && (9)\end{aligned}$$

If $c_a(-\infty) = 1$ and $c_b(-\infty) = 0$ then, since the perturbation is switched off until $t = 0$ these coefficients remain constant until that time, so we must have $c_a(0) = 1$ and $c_b(0) = 0$. This means that this system is the same as that we treated in the last post, where a perturbation is switched on at $t = 0$ and off at a later time, so the coefficients are

$$c_a(t) = e^{-i\omega_0 t/2}\left[\cos(Qt) + \frac{i\omega_0}{2Q}\sin(Qt)\right] \tag{10}$$

$$\begin{aligned} c_b(t) &= -\frac{i\left|H'_{ab}\right|}{\hbar Q} e^{i\omega_0 t/2}\sin(Qt) && (11)\\ &= -\frac{i\alpha}{\hbar Q\tau} e^{i\omega_0 t/2}\sin(Qt) && (12)\end{aligned}$$

where

$$\begin{aligned} Q &\equiv \frac{1}{2}\sqrt{\omega_0^2 + \frac{4\left|H'_{ab}\right|^2}{\hbar^2}} && (13)\\ &= \frac{1}{2}\sqrt{\omega_0^2 + \frac{4\alpha^2}{\hbar^2\tau^2}} && (14)\\ \omega_0 &\equiv \frac{E_b - E_a}{\hbar} && (15)\end{aligned}$$

Since the perturbation is switched off at $t = \tau$, c_a and c_b with remain constant from $t = \tau$ onwards.

Since this is a special case of the earlier problem, we have $|c_a|^2 + |c_b|^2 = 1$ here as well.

To get a delta function perturbation, we let $\tau \to 0$. In this limit, we get

$$Q \sim \frac{|\alpha|}{\hbar\tau} \tag{16}$$

and the coefficients become

$$c_a(\tau) = e^{-i\omega_0\tau/2}\left[\cos(Q\tau) + \frac{i\omega_0}{2Q}\sin(Q\tau)\right] \tag{17}$$

$$\sim \cos\left(\frac{|\alpha|}{\hbar}\right) \tag{18}$$

$$c_b(\tau) = -\frac{i\alpha}{\hbar Q\tau}e^{i\omega_0\tau/2}\sin(Q\tau) \tag{19}$$

$$\sim \pm i\sin\left(\frac{|\alpha|}{\hbar}\right) \tag{20}$$

where the $\pm$ in the last line arises because of the quotient $\alpha/|\alpha|$. Since the system starts out in state a (because $c_a(0) = 1$), the probability that it has made a transition to state b after the perturbation is switched off is

$$P_{a\to b} = |c_b(\infty)|^2 = |c_b(\tau)|^2 = \sin^2\left(\frac{|\alpha|}{\hbar}\right) \tag{21}$$