DELTA FUNCTION IN TIME PERTURBATION

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 9.3.

We've seen that we can solve the Schrödinger equation with a time-dependent potential in a two-state system if we split the hamiltonian into a time-independent part H^0 and a time-dependent part H', so that the complete hamiltonian is

$$H = H^0 + H' \tag{0.1}$$

The solution is

$$\Psi(x,t) = c_a(t)\,\psi_a(x)\,e^{-iE_at/\hbar} + c_b(t)\,\psi_b(x)\,e^{-iE_bt/\hbar} \tag{0.2}$$

As an application of the situation where a perturbation to the hamiltonian in a two-state system is switched on at $t=0$ and then off again at some later time $t=\tau$ we can look at a perturbation that is on only momentarily, which we can model by using a delta function:

$$H' = U(x)\,\delta(t) \tag{0.3}$$

We can think of this as the limit as $\tau \to 0$ of the earlier problem if we start with

$$H' = \begin{cases} \frac{U}{\tau} & 0 \le t \le \tau \\ 0 & \text{otherwise} \end{cases} \tag{0.4}$$

This works because the integral

$$\int_0^\tau \frac{dt}{\tau} = 1 \tag{0.5}$$

so as $\tau \to 0$ 0.4 becomes 0.3.

If we assume that the matrix elements of H' in the two-state system satisfy

$$U_{aa} = U_{bb} = 0 \tag{0.6}$$
$$U_{ab} = U_{ba}^{*} \equiv \alpha \tag{0.7}$$

then in the interval $0 \le t \le \tau$ we have

$$H'_{aa} = H'_{bb} = 0 \tag{0.8}$$
$$H'_{ab} = \left(H'_{ba}\right)^{*} = \frac{\alpha}{\tau} \tag{0.9}$$

If $c_a(-\infty) = 1$ and $c_b(-\infty) = 0$ then, since the perturbation is switched off until $t = 0$ these coefficients remain constant until that time, so we must have $c_a(0) = 1$ and $c_b(0) = 0$. This means that this system is the same as that we treated in the last post, where a perturbation is switched on at $t = 0$ and off at a later time, so the coefficients are

$$c_a(t) = e^{-i\omega_0 t/2}\left[\cos(Qt) + \frac{i\omega_0}{2Q}\sin(Qt)\right] \tag{0.10}$$
$$c_b(t) = -\frac{i\left|H'_{ab}\right|}{\hbar Q}e^{i\omega_0 t/2}\sin(Qt) \tag{0.11}$$
$$= -\frac{i\alpha}{\hbar Q\tau}e^{i\omega_0 t/2}\sin(Qt) \tag{0.12}$$

where

$$Q \equiv \frac{1}{2}\sqrt{\omega_0^2 + \frac{4\left|H'_{ab}\right|^2}{\hbar^2}} \tag{0.13}$$
$$= \frac{1}{2}\sqrt{\omega_0^2 + \frac{4\alpha^2}{\hbar^2\tau^2}} \tag{0.14}$$
$$\omega_0 \equiv \frac{E_b - E_a}{\hbar} \tag{0.15}$$

Since the perturbation is switched off at $t = \tau$, c_a and c_b with remain constant from $t = \tau$ onwards.

Since this is a special case of the earlier problem, we have $|c_a|^2 + |c_b|^2 = 1$ here as well.

To get a delta function perturbation, we let $\tau \to 0$. In this limit, we get

$$Q \sim \frac{|\alpha|}{\hbar\tau} \tag{0.16}$$

and the coefficients become

$$c_a(\tau) = e^{-i\omega_0\tau/2}\left[\cos(Q\tau) + \frac{i\omega_0}{2Q}\sin(Q\tau)\right] \tag{0.17}$$

$$\sim \cos\left(\frac{|\alpha|}{\hbar}\right) \tag{0.18}$$

$$c_b(\tau) = -\frac{i\alpha}{\hbar Q\tau}e^{i\omega_0\tau/2}\sin(Q\tau) \tag{0.19}$$

$$\sim \pm i\sin\left(\frac{|\alpha|}{\hbar}\right) \tag{0.20}$$

where the $\pm$ in the last line arises because of the quotient $\alpha/|\alpha|$. Since the system starts out in state a (because $c_a(0) = 1$), the probability that it has made a transition to state b after the perturbation is switched off is

$$P_{a\to b} = |c_b(\infty)|^2 = |c_b(\tau)|^2 = \sin^2\left(\frac{|\alpha|}{\hbar}\right) \tag{0.21}$$