DELTA FUNCTION IN TIME PERTURBATION

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We’ve seen that we can solve the Schrödinger equation with a time-dependent potential in a two-state system if we split the Hamiltonian into a time-independent part $H^0$ and a time-dependent part $H'$, so that the complete Hamiltonian is

$$H = H^0 + H'$$  \hspace{1cm} (1)$$

The solution is

$$\Psi(x,t) = c_a(t) \psi_a(x) e^{-iE_a t/\hbar} + c_b(t) \psi_b(x) e^{-iE_b t/\hbar}$$  \hspace{1cm} (2)$$

As an application of the situation where a perturbation to the Hamiltonian in a two-state system is switched on at $t = 0$ and then off again at some later time $t = \tau$ we can look at a perturbation that is on only momentarily, which we can model by using a delta function:

$$H' = U(x) \delta(t)$$  \hspace{1cm} (3)$$

We can think of this as the limit as $\tau \to 0$ of the earlier problem if we start with

$$H' = \begin{cases} U \frac{t}{\tau} & 0 \leq t \leq \tau \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (4)$$

This works because the integral

$$\int_0^\tau \frac{dt}{\tau} = 1$$  \hspace{1cm} (5)$$

so as $\tau \to 0$ becomes

If we assume that the matrix elements of $H'$ in the two-state system satisfy

$$U_{aa} = U_{bb} = 0$$  \hspace{1cm} (6)$$

$$U_{ab} = U_{ba} = \alpha$$  \hspace{1cm} (7)$$
then in the interval \(0 \leq t \leq \tau\) we have

\[
H'_{aa} = H'_{bb} = 0 \quad (8)
\]

\[
H'_{ab} = (H'_{ba})^* = \frac{\alpha}{\tau} \quad (9)
\]

If \(c_a(-\infty) = 1\) and \(c_b(-\infty) = 0\) then, since the perturbation is switched off until \(t = 0\) these coefficients remain constant until that time, so we must have \(c_a(0) = 1\) and \(c_b(0) = 0\). This means that this system is the same as that we treated in the last post, where a perturbation is switched on at \(t = 0\) and off at a later time, so the coefficients are

\[
c_a(t) = e^{-i\omega_0 t/2} \left[ \cos(Qt) + \frac{i\omega_0}{2Q} \sin(Qt) \right] \quad (10)
\]

\[
c_b(t) = -\frac{i}{\hbar Q} e^{i\omega_0 t/2} \sin(Qt) \quad (11)
\]

\[
c_b(t) = -\frac{i\alpha}{\hbar Q \tau} e^{i\omega_0 t/2} \sin(Qt) \quad (12)
\]

where

\[
Q \equiv \frac{1}{2} \sqrt{\omega_0^2 + \frac{4|H'_{ab}|^2}{\hbar^2}} \quad (13)
\]

\[
= \frac{1}{2} \sqrt{\omega_0^2 + \frac{4\alpha^2}{\hbar^2 \tau^2}} \quad (14)
\]

\[
\omega_0 \equiv \frac{E_b - E_a}{\hbar} \quad (15)
\]

Since the perturbation is switched off at \(t = \tau\), \(c_a\) and \(c_b\) with remain constant from \(t = \tau\) onwards.

Since this is a special case of the earlier problem, we have \(|c_a|^2 + |c_b|^2 = 1\) here as well.

To get a delta function perturbation, we let \(\tau \to 0\). In this limit, we get

\[
Q \sim \frac{|\alpha|}{\hbar \tau} \quad (16)
\]

and the coefficients become
\[ c_a(\tau) = e^{-i\omega_0 \tau/2} \left[ \cos (Q\tau) + \frac{i\omega_0}{2Q} \sin (Q\tau) \right] \]  \hspace{1cm} (17)

\[ \sim \cos \left( \frac{|\alpha|}{\hbar} \right) \]  \hspace{1cm} (18)

\[ c_b(\tau) = -\frac{i\alpha}{\hbar Q} e^{i\omega_0 \tau/2} \sin (Q\tau) \]  \hspace{1cm} (19)

\[ \sim \pm i \sin \left( \frac{|\alpha|}{\hbar} \right) \]  \hspace{1cm} (20)

where the \pm in the last line arises because of the quotient \( \alpha/|\alpha| \). Since the system starts out in state \( a \) (because \( c_a(0) = 1 \)), the probability that it has made a transition to state \( b \) after the perturbation is switched off is

\[ P_{a\rightarrow b} = |c_b(\infty)|^2 = |c_b(\tau)|^2 = \sin^2 \left( \frac{|\alpha|}{\hbar} \right) \]  \hspace{1cm} (21)