

## TIME-DEPENDENT PERTURBATION THEORY: GENERAL TWO-STATE SOLUTION

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 9.5.

We've seen that we can solve the Schrödinger equation with a time-dependent potential in a two-state system if we split the hamiltonian into a time-independent part  $H^0$  and a time-dependent part  $H'$ , so that the complete hamiltonian is

$$H = H^0 + H' \quad (1)$$

The solution is

$$\Psi(x, t) = c_a(t) \psi_a(x) e^{-iE_a t/\hbar} + c_b(t) \psi_b(x) e^{-iE_b t/\hbar} \quad (2)$$

where  $\psi_a$  and  $\psi_b$  are the two eigenstates of  $H^0$  and the coefficients are solutions of the coupled ODEs

$$\dot{c}_a = -\frac{i}{\hbar} \left[ c_a H'_{aa} + c_b H'_{ab} e^{-i(E_b - E_a)t/\hbar} \right] \quad (3)$$

$$\dot{c}_b = -\frac{i}{\hbar} \left[ c_b H'_{bb} + c_a H'_{ba} e^{i(E_b - E_a)t/\hbar} \right] \quad (4)$$

where

$$H'_{ij} \equiv \langle \psi_i | H' | \psi_j \rangle \quad (5)$$

If we make the assumption that  $H'$  is small and that the diagonal elements  $H'_{aa} = H'_{bb} = 0$ , then we can apply the iterative method of time-dependent perturbation theory to work out the coefficients  $c_a$  and  $c_b$  in the general case where the initial conditions are

$$c_a(0) = a \quad (6)$$

$$c_b(0) = b \quad (7)$$

These conditions form the zeroth-order solution, so we plug them into 3 and 4 with zero diagonal elements to get

$$\dot{c}_a^{(1)} = -\frac{i}{\hbar} b H'_{ab} e^{-i(E_b - E_a)t/\hbar} \quad (8)$$

$$c_a^{(1)}(t) = a - \frac{ib}{\hbar} \int_0^t H'_{ab} e^{-i(E_b - E_a)t'/\hbar} dt' \quad (9)$$

$$\dot{c}_b^{(1)} = -\frac{i}{\hbar} a H'_{ba}(t') e^{i(E_b - E_a)t/\hbar} \quad (10)$$

$$c_b^{(1)}(t) = b - \frac{ia}{\hbar} \int_0^t H'_{ba}(t') e^{i(E_b - E_a)t'/\hbar} dt' \quad (11)$$

Continuing to second order, we get

$$\dot{c}_a^{(2)} = -\frac{i}{\hbar} \left( b - \frac{ia}{\hbar} \int_0^t H'_{ba}(t') e^{i(E_b - E_a)t'/\hbar} dt' \right) H'_{ab} e^{-i(E_b - E_a)t/\hbar} \quad (12)$$

$$c_a^{(2)}(t) = a - \frac{ib}{\hbar} \int_0^t H'_{ab} e^{-i(E_b - E_a)t'/\hbar} dt' - \quad (13)$$

$$\frac{a}{\hbar^2} \int_0^t H'_{ab}(t') e^{-i(E_b - E_a)t'/\hbar} \left[ \int_0^{t'} H'_{ba}(t'') e^{i(E_b - E_a)t''/\hbar} dt'' \right] dt'$$

$$\dot{c}_b^{(2)} = -\frac{i}{\hbar} \left( a - \frac{ib}{\hbar} \int_0^t H'_{ab} e^{-i(E_b - E_a)t'/\hbar} dt' \right) H'_{ba}(t') e^{i(E_b - E_a)t/\hbar} \quad (14)$$

$$c_b^{(2)}(t) = b - \frac{ia}{\hbar} \int_0^t H'_{ba}(t') e^{i(E_b - E_a)t'/\hbar} dt' - \quad (15)$$

$$\frac{b}{\hbar^2} \int_0^t H'_{ba}(t') e^{i(E_b - E_a)t'/\hbar} \left[ \int_0^{t'} H'_{ab}(t'') e^{-i(E_b - E_a)t''/\hbar} dt'' \right] dt'$$