# TIME-DEPENDENT PERTURBATION THEORY: GENERAL TWO-STATE SOLUTION

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 9.5.

We've seen that we can solve the Schrödinger equation with a time-dependent potential in a two-state system if we split the hamiltonian into a time-independent part $H^0$ and a time-dependent part $H'$, so that the complete hamiltonian is

$$H = H^0 + H' \tag{1}$$

The solution is

$$\Psi(x,t) = c_a(t)\,\psi_a(x)\,e^{-iE_at/\hbar} + c_b(t)\,\psi_b(x)\,e^{-iE_bt/\hbar} \tag{2}$$

where $\psi_a$ and $\psi_b$ are the two eigenstates of $H^0$ and the coefficients are solutions of the coupled ODEs

$$\dot{c}_a = -\frac{i}{\hbar}\left[c_aH'_{aa} + c_bH'_{ab}e^{-i(E_b-E_a)t/\hbar}\right] \tag{3}$$

$$\dot{c}_b = -\frac{i}{\hbar}\left[c_bH'_{bb} + c_aH'_{ba}e^{i(E_b-E_a)t/\hbar}\right] \tag{4}$$

where

$$H'_{ij} \equiv \langle\psi_i|H'\left|\psi_j\right\rangle \tag{5}$$

If we make the assumption that $H'$ is small and that the diagonal elements $H'_{aa} = H'_{bb} = 0$, then we can apply the iterative method of time-dependent perturbation theory to work out the coefficients $c_a$ and $c_b$ in the general case where the initial conditions are

$$c_a(0) = a \tag{6}$$

$$c_b(0) = b \tag{7}$$

These conditions form the zeroth-order solution, so we plug them into 3 and 4 with zero diagonal elements to get

$$\dot{c}_a^{(1)} = -\frac{i}{\hbar} b H'_{ab} e^{-i(E_b - E_a)t/\hbar} \tag{8}$$

$$c_a^{(1)}(t) = a - \frac{ib}{\hbar} \int_0^t H'_{ab} e^{-i(E_b - E_a)t'/\hbar} dt' \tag{9}$$

$$\dot{c}_b^{(1)} = -\frac{i}{\hbar} a H'_{ba}(t') e^{i(E_b - E_a)t/\hbar} \tag{10}$$

$$c_b^{(1)}(t) = b - \frac{ia}{\hbar} \int_0^t H'_{ba}(t') e^{i(E_b - E_a)t'/\hbar} dt' \tag{11}$$

Continuing to second order, we get

$$\dot{c}_a^{(2)} = -\frac{i}{\hbar} \left( b - \frac{ia}{\hbar} \int_0^t H'_{ba}(t') e^{i(E_b - E_a)t'/\hbar} dt' \right) H'_{ab} e^{-i(E_b - E_a)t/\hbar} \tag{12}$$

$$c_a^{(2)}(t) = a - \frac{ib}{\hbar} \int_0^t H'_{ab} e^{-i(E_b - E_a)t'/\hbar} dt' - \frac{a}{\hbar^2} \int_0^t H'_{ab}(t') e^{-i(E_b - E_a)t'/\hbar} \left[ \int_0^{t'} H'_{ba}(t'') e^{i(E_b - E_a)t''/\hbar} dt'' \right] dt' \tag{13}$$

$$\dot{c}_b^{(2)} = -\frac{i}{\hbar} \left( a - \frac{ib}{\hbar} \int_0^t H'_{ab} e^{-i(E_b - E_a)t'/\hbar} dt' \right) H'_{ba}(t') e^{i(E_b - E_a)t/\hbar} \tag{14}$$

$$c_b^{(2)}(t) = b - \frac{ia}{\hbar} \int_0^t H'_{ba}(t') e^{i(E_b - E_a)t'/\hbar} dt' - \frac{b}{\hbar^2} \int_0^t H'_{ba}(t') e^{i(E_b - E_a)t'/\hbar} \left[ \int_0^{t'} H'_{ab}(t'') e^{-i(E_b - E_a)t''/\hbar} dt'' \right] dt' \tag{15}$$