TIME-DEPENDENT PERTURBATION THEORY: SWITCHING A PERTURBATION ON AND OFF

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 9.6.

Earlier, we analyzed the case of a constant (in time) perturbation that is switched on at $t=0$ and off again at $t=\tau$. That analysis was exact, as it did not make any assumptions about the perturbation being small. We can apply time-dependent perturbation theory to get estimates of the coefficients in the solution

$$\Psi(x,t)=c_a(t)\,\psi_a(x)\,e^{-iE_at/\hbar}+c_b(t)\,\psi_b(x)\,e^{-iE_bt/\hbar} \tag{1}$$

where the complete hamiltonian is $H=H^0+H'$, ψ_a and ψ_b are the two eigenstates of H^0 and the coefficients are solutions of the coupled ODEs

$$\dot{c}_a=-\frac{i}{\hbar}\left[c_aH'_{aa}+c_bH'_{ab}e^{-i(E_b-E_a)t/\hbar}\right] \tag{2}$$

$$\dot{c}_b=-\frac{i}{\hbar}\left[c_bH'_{bb}+c_aH'_{ba}e^{i(E_b-E_a)t/\hbar}\right] \tag{3}$$

where

$$H'_{ij}\equiv\langle\psi_i|\,H'\,|\psi_j\rangle \tag{4}$$

We can start with the general solution for the second order terms:

$$\begin{aligned} c_a^{(2)}(t)=a-\frac{ib}{\hbar}\int_0^t H'_{ab}(t')\,e^{-i(E_b-E_a)t'/\hbar}dt'- \\ \frac{a}{\hbar^2}\int_0^t H'_{ab}(t')\,e^{-i(E_b-E_a)t'/\hbar}\left[\int_0^{t'}H'_{ba}(t'')\,e^{i(E_b-E_a)t''/\hbar}dt''\right]dt' \end{aligned} \tag{5}$$

$$\begin{aligned} c_b^{(2)}(t)=b-\frac{ia}{\hbar}\int_0^t H'_{ba}(t')\,e^{i(E_b-E_a)t'/\hbar}dt'- \\ \frac{b}{\hbar^2}\int_0^t H'_{ba}(t')\,e^{i(E_b-E_a)t'/\hbar}\left[\int_0^{t'}H'_{ab}(t'')\,e^{-i(E_b-E_a)t''/\hbar}dt''\right]dt' \end{aligned} \tag{6}$$

Since H' is independent of time, it can come outside the integral and since $\left(H'_{ab}\right)^* = H'_{ba}$ we have (where $c_a(0) = a$ and $c_b(0) = b$):

$$c_a^{(2)}(t) = a - \frac{ib}{\hbar} H'_{ab} \int_0^t e^{-i(E_b-E_a)t'/\hbar} dt' - \tag{7}$$

$$\frac{a}{\hbar^2} \left|H'_{ab}\right|^2 \int_0^t \int_0^{t'} e^{-i(E_b-E_a)(t''-t')/\hbar} dt'' dt'$$

$$= a + \frac{bH'_{ab}}{E_b - E_a} \left(e^{-i(E_b-E_a)t/\hbar} - 1\right) + \tag{8}$$

$$\frac{a\left|H'_{ab}\right|^2}{\left(E_b - E_a\right)^2} \left(e^{-i(E_b-E_a)t/\hbar} + \frac{i}{\hbar}\left(E_b - E_a\right)t - 1\right)$$

$$c_b^{(2)}(t) = b - \frac{ia}{\hbar} H'_{ba} \int_0^t e^{i(E_b-E_a)t'/\hbar} dt' - \tag{9}$$

$$\frac{b}{\hbar^2} \left|H'_{ab}\right|^2 \int_0^t \int_0^{t'} e^{-i(E_b-E_a)(t''-t')/\hbar} dt'' dt'$$

$$= b - \frac{aH'_{ba}}{E_b - E_a} \left(e^{i(E_b-E_a)t/\hbar} - 1\right) - \tag{10}$$

$$\frac{b\left|H'_{ab}\right|^2}{\left(E_b - E_a\right)^2} \left(e^{i(E_b-E_a)t/\hbar} - \frac{i}{\hbar}\left(E_b - E_a\right)t - 1\right)$$

To compare these results with the exact solution we worked out earlier, we set the initial conditions $a = 1$ and $b = 0$, and define $\omega_0 \equiv \left(E_b - E_a\right)/\hbar$. Then

$$c_a^{(2)}(t) = 1 + \frac{\left|H'_{ab}\right|^2}{\hbar^2 \omega_0^2} \left(e^{-i\omega_0 t} + i\omega_0 t - 1\right) \tag{11}$$

$$c_b^{(2)}(t) = -\frac{H'_{ba}}{\hbar\omega_0} \left(e^{i\omega_0 t} - 1\right) \tag{12}$$

Our earlier exact formulas are

$$c_a(t) = e^{-i\omega_0 t/2} \left[\cos(Qt) + \frac{i\omega_0}{2Q} \sin(Qt)\right] \tag{13}$$

$$c_b(t) = -\frac{i\left|H'_{ab}\right|}{\hbar Q} e^{i\omega_0 t/2} \sin(Qt) \tag{14}$$

where

$$Q \equiv \frac{1}{2}\sqrt{\omega_0^2 + \frac{4\left|H'_{ab}\right|^2}{\hbar^2}} \tag{15}$$

To compare these two sets of formulas, we need to Taylor expand the latter pair up to second order in H'. This is a messy procedure and is best consigned to Maple, from which we get

$$c_a(t) = 1 + \frac{\left|H'_{ab}\right|^2}{\hbar^2\omega_0^2}\left(e^{-i\omega_0 t} + i\omega_0 t - 1\right) + \mathcal{O}\left(\left|H'\right|^4\right) \tag{16}$$

$$c_b(t) = -\frac{2i\left|H'_{ab}\right|}{\hbar\omega_0}e^{i\omega_0 t/2}\sin\frac{\omega_0 t}{2} + \mathcal{O}\left(\left|H'\right|^3\right) \tag{17}$$

$$= -\frac{2i\left|H'_{ab}\right|}{\hbar\omega_0}e^{i\omega_0 t/2}\frac{1}{2i}\left(e^{i\omega_0 t/2} - e^{-i\omega_0 t/2}\right) + \mathcal{O}\left(\left|H'\right|^3\right) \tag{18}$$

$$= -\frac{H'_{ba}}{\hbar\omega_0}\left(e^{i\omega_0 t} - 1\right) + \mathcal{O}\left(\left|H'\right|^3\right) \tag{19}$$

Thus the formulas agree up to second order.