

TIME-DEPENDENT PERTURBATION THEORY: SWITCHING A PERTURBATION ON AND OFF

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 9.6.

Earlier, we analyzed the case of a constant (in time) perturbation that is switched on at $t = 0$ and off again at $t = \tau$. That analysis was exact, as it did not make any assumptions about the perturbation being small. We can apply time-dependent perturbation theory to get estimates of the coefficients in the solution

$$(1) \quad \Psi(x, t) = c_a(t) \psi_a(x) e^{-iE_a t/\hbar} + c_b(t) \psi_b(x) e^{-iE_b t/\hbar}$$

where the complete hamiltonian is $H = H^0 + H'$, ψ_a and ψ_b are the two eigenstates of H^0 and the coefficients are solutions of the coupled ODEs

$$(2) \quad \dot{c}_a = -\frac{i}{\hbar} \left[c_a H'_{aa} + c_b H'_{ab} e^{-i(E_b - E_a)t/\hbar} \right]$$

$$(3) \quad \dot{c}_b = -\frac{i}{\hbar} \left[c_b H'_{bb} + c_a H'_{ba} e^{i(E_b - E_a)t/\hbar} \right]$$

where

$$(4) \quad H'_{ij} \equiv \langle \psi_i | H' | \psi_j \rangle$$

We can start with the general solution for the second order terms:

$$(5) \quad c_a^{(2)}(t) = a - \frac{ib}{\hbar} \int_0^t H'_{ab}(t') e^{-i(E_b-E_a)t'/\hbar} dt' - \frac{a}{\hbar^2} \int_0^t H'_{ab}(t') e^{-i(E_b-E_a)t'/\hbar} \left[\int_0^{t'} H'_{ba}(t'') e^{i(E_b-E_a)t''/\hbar} dt'' \right] dt'$$

$$(6) \quad c_b^{(2)}(t) = b - \frac{ia}{\hbar} \int_0^t H'_{ba}(t') e^{i(E_b-E_a)t'/\hbar} dt' - \frac{b}{\hbar^2} \int_0^t H'_{ba}(t') e^{i(E_b-E_a)t'/\hbar} \left[\int_0^{t'} H'_{ab}(t'') e^{-i(E_b-E_a)t''/\hbar} dt'' \right] dt'$$

Since H' is independent of time, it can come outside the integral and since $(H'_{ab})^* = H'_{ba}$ we have (where $c_a(0) = a$ and $c_b(0) = b$):

$$(7) \quad c_a^{(2)}(t) = a - \frac{ib}{\hbar} H'_{ab} \int_0^t e^{-i(E_b-E_a)t'/\hbar} dt' - \frac{a}{\hbar^2} |H'_{ab}|^2 \int_0^t \int_0^{t'} e^{-i(E_b-E_a)(t''-t')/\hbar} dt'' dt'$$

$$(8) \quad = a + \frac{bH'_{ab}}{E_b - E_a} \left(e^{-i(E_b-E_a)t/\hbar} - 1 \right) + \frac{a|H'_{ab}|^2}{(E_b - E_a)^2} \left(e^{-i(E_b-E_a)t/\hbar} + \frac{i}{\hbar} (E_b - E_a)t - 1 \right)$$

$$(9) \quad c_b^{(2)}(t) = b - \frac{ia}{\hbar} H'_{ba} \int_0^t e^{i(E_b-E_a)t'/\hbar} dt' - \frac{b}{\hbar^2} |H'_{ab}|^2 \int_0^t \int_0^{t'} e^{-i(E_b-E_a)(t''-t')/\hbar} dt'' dt'$$

$$(10) \quad = b - \frac{aH'_{ba}}{E_b - E_a} \left(e^{i(E_b-E_a)t/\hbar} - 1 \right) - \frac{b|H'_{ab}|^2}{(E_b - E_a)^2} \left(e^{i(E_b-E_a)t/\hbar} - \frac{i}{\hbar} (E_b - E_a)t - 1 \right)$$

To compare these results with the exact solution we worked out earlier, we set the initial conditions $a = 1$ and $b = 0$, and define $\omega_0 \equiv (E_b - E_a)/\hbar$. Then

$$(11) \quad c_a^{(2)}(t) = 1 + \frac{|H'_{ab}|^2}{\hbar^2 \omega_0^2} (e^{-i\omega_0 t} + i\omega_0 t - 1)$$

$$(12) \quad c_b^{(2)}(t) = -\frac{H'_{ba}}{\hbar \omega_0} (e^{i\omega_0 t} - 1)$$

Our earlier exact formulas are

$$(13) \quad c_a(t) = e^{-i\omega_0 t/2} \left[\cos(Qt) + \frac{i\omega_0}{2Q} \sin(Qt) \right]$$

$$(14) \quad c_b(t) = -\frac{i|H'_{ab}|}{\hbar Q} e^{i\omega_0 t/2} \sin(Qt)$$

where

$$(15) \quad Q \equiv \frac{1}{2} \sqrt{\omega_0^2 + \frac{4|H'_{ab}|^2}{\hbar^2}}$$

To compare these two sets of formulas, we need to Taylor expand the latter pair up to second order in H' . This is a messy procedure and is best consigned to Maple, from which we get

$$(16) \quad c_a(t) = 1 + \frac{|H'_{ab}|^2}{\hbar^2 \omega_0^2} (e^{-i\omega_0 t} + i\omega_0 t - 1) + \mathcal{O}(|H'|^4)$$

$$(17) \quad c_b(t) = -\frac{2i|H'_{ab}|}{\hbar \omega_0} e^{i\omega_0 t/2} \sin \frac{\omega_0 t}{2} + \mathcal{O}(|H'|^3)$$

$$(18) \quad = -\frac{2i|H'_{ab}|}{\hbar \omega_0} e^{i\omega_0 t/2} \frac{1}{2i} (e^{i\omega_0 t/2} - e^{-i\omega_0 t/2}) + \mathcal{O}(|H'|^3)$$

$$(19) \quad = -\frac{H'_{ba}}{\hbar \omega_0} (e^{i\omega_0 t} - 1) + \mathcal{O}(|H'|^3)$$

Thus the formulas agree up to second order.