TIME-DEPENDENT PERTURBATION THEORY: SWITCHING A PERTURBATION ON AND OFF

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 9.6.

Earlier, we analyzed the case of a constant (in time) perturbation that is switched on at $t=0$ and off again at $t=\tau$. That analysis was exact, as it did not make any assumptions about the perturbation being small. We can apply time-dependent perturbation theory to get estimates of the coefficients in the solution

$$\Psi(x,t)=c_a(t)\,\psi_a(x)\,e^{-iE_at/\hbar}+c_b(t)\,\psi_b(x)\,e^{-iE_bt/\hbar} \tag{0.1}$$

where the complete hamiltonian is $H=H^0+H'$, ψ_a and ψ_b are the two eigenstates of H^0 and the coefficients are solutions of the coupled ODEs

$$\dot{c}_a=-\frac{i}{\hbar}\left[c_aH'_{aa}+c_bH'_{ab}e^{-i(E_b-E_a)t/\hbar}\right] \tag{0.2}$$

$$\dot{c}_b=-\frac{i}{\hbar}\left[c_bH'_{bb}+c_aH'_{ba}e^{i(E_b-E_a)t/\hbar}\right] \tag{0.3}$$

where

$$H'_{ij}\equiv\langle\psi_i|H'\left|\psi_j\right\rangle \tag{0.4}$$

We can start with the general solution for the second order terms:

(0.5)

$$c_a^{(2)}(t) = a - \frac{ib}{\hbar}\int_0^t H'_{ab}(t')\, e^{-i(E_b-E_a)t'/\hbar} dt' - \frac{a}{\hbar^2}\int_0^t H'_{ab}(t')\, e^{-i(E_b-E_a)t'/\hbar}\left[\int_0^{t'} H'_{ba}(t'')\, e^{i(E_b-E_a)t''/\hbar} dt''\right] dt'$$

(0.6)

$$c_b^{(2)}(t) = b - \frac{ia}{\hbar}\int_0^t H'_{ba}(t')\, e^{i(E_b-E_a)t'/\hbar} dt' - \frac{b}{\hbar^2}\int_0^t H'_{ba}(t')\, e^{i(E_b-E_a)t'/\hbar}\left[\int_0^{t'} H'_{ab}(t'')\, e^{-i(E_b-E_a)t''/\hbar} dt''\right] dt'$$

Since H' is independent of time, it can come outside the integral and since $\left(H'_{ab}\right)^* = H'_{ba}$ we have (where $c_a(0) = a$ and $c_b(0) = b$):

$$c_a^{(2)}(t) = a - \frac{ib}{\hbar}H'_{ab}\int_0^t e^{-i(E_b-E_a)t'/\hbar} dt' - \frac{a}{\hbar^2}\left|H'_{ab}\right|^2 \int_0^t \int_0^{t'} e^{-i(E_b-E_a)(t''-t')/\hbar} dt'' dt' \tag{0.7}$$

$$= a + \frac{bH'_{ab}}{E_b-E_a}\left(e^{-i(E_b-E_a)t/\hbar} - 1\right) + \frac{a\left|H'_{ab}\right|^2}{(E_b-E_a)^2}\left(e^{-i(E_b-E_a)t/\hbar} + \frac{i}{\hbar}(E_b-E_a)t - 1\right) \tag{0.8}$$

$$c_b^{(2)}(t) = b - \frac{ia}{\hbar}H'_{ba}\int_0^t e^{i(E_b-E_a)t'/\hbar} dt' - \frac{b}{\hbar^2}\left|H'_{ab}\right|^2 \int_0^t \int_0^{t'} e^{-i(E_b-E_a)(t''-t')/\hbar} dt'' dt' \tag{0.9}$$

$$= b - \frac{aH'_{ba}}{E_b-E_a}\left(e^{i(E_b-E_a)t/\hbar} - 1\right) - \frac{b\left|H'_{ab}\right|^2}{(E_b-E_a)^2}\left(e^{i(E_b-E_a)t/\hbar} - \frac{i}{\hbar}(E_b-E_a)t - 1\right) \tag{0.10}$$

To compare these results with the exact solution we worked out earlier, we set the initial conditions $a = 1$ and $b = 0$, and define $\omega_0 \equiv (E_b - E_a)/\hbar$. Then

$$c_a^{(2)}(t) = 1 + \frac{\left|H'_{ab}\right|^2}{\hbar^2\omega_0^2}\left(e^{-i\omega_0 t} + i\omega_0 t - 1\right) \tag{0.11}$$

$$c_b^{(2)}(t) = -\frac{H'_{ba}}{\hbar\omega_0}\left(e^{i\omega_0 t} - 1\right) \tag{0.12}$$

Our earlier exact formulas are

$$c_a(t) = e^{-i\omega_0 t/2}\left[\cos(Qt) + \frac{i\omega_0}{2Q}\sin(Qt)\right] \tag{0.13}$$

$$c_b(t) = -\frac{i\left|H'_{ab}\right|}{\hbar Q}e^{i\omega_0 t/2}\sin(Qt) \tag{0.14}$$

where

$$Q \equiv \frac{1}{2}\sqrt{\omega_0^2 + \frac{4\left|H'_{ab}\right|^2}{\hbar^2}} \tag{0.15}$$

To compare these two sets of formulas, we need to Taylor expand the latter pair up to second order in H'. This is a messy procedure and is best consigned to Maple, from which we get

$$c_a(t) = 1 + \frac{\left|H'_{ab}\right|^2}{\hbar^2\omega_0^2}\left(e^{-i\omega_0 t} + i\omega_0 t - 1\right) + \mathscr{O}\left(\left|H'\right|^4\right) \tag{0.16}$$

$$c_b(t) = -\frac{2i\left|H'_{ab}\right|}{\hbar\omega_0}e^{i\omega_0 t/2}\sin\frac{\omega_0 t}{2} + \mathscr{O}\left(\left|H'\right|^3\right) \tag{0.17}$$

$$= -\frac{2i\left|H'_{ab}\right|}{\hbar\omega_0}e^{i\omega_0 t/2}\frac{1}{2i}\left(e^{i\omega_0 t/2} - e^{-i\omega_0 t/2}\right) + \mathscr{O}\left(\left|H'\right|^3\right) \tag{0.18}$$

$$= -\frac{H'_{ba}}{\hbar\omega_0}\left(e^{i\omega_0 t} - 1\right) + \mathscr{O}\left(\left|H'\right|^3\right) \tag{0.19}$$

Thus the formulas agree up to second order.