

## TIME-DEPENDENT PERTURBATION THEORY: SWITCHING A PERTURBATION ON AND OFF

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 9.6.

Earlier, we analyzed the case of a constant (in time) perturbation that is switched on at  $t = 0$  and off again at  $t = \tau$ . That analysis was exact, as it did not make any assumptions about the perturbation being small. We can apply time-dependent perturbation theory to get estimates of the coefficients in the solution

$$\Psi(x, t) = c_a(t) \psi_a(x) e^{-iE_a t/\hbar} + c_b(t) \psi_b(x) e^{-iE_b t/\hbar} \quad (1)$$

where the complete hamiltonian is  $H = H^0 + H'$ ,  $\psi_a$  and  $\psi_b$  are the two eigenstates of  $H^0$  and the coefficients are solutions of the coupled ODEs

$$\dot{c}_a = -\frac{i}{\hbar} \left[ c_a H'_{aa} + c_b H'_{ab} e^{-i(E_b - E_a)t/\hbar} \right] \quad (2)$$

$$\dot{c}_b = -\frac{i}{\hbar} \left[ c_b H'_{bb} + c_a H'_{ba} e^{i(E_b - E_a)t/\hbar} \right] \quad (3)$$

where

$$H'_{ij} \equiv \langle \psi_i | H' | \psi_j \rangle \quad (4)$$

We can start with the general solution for the second order terms:

$$c_a^{(2)}(t) = a - \frac{ib}{\hbar} \int_0^t H'_{ab}(t') e^{-i(E_b - E_a)t'/\hbar} dt' - \quad (5)$$

$$\frac{a}{\hbar^2} \int_0^t H'_{ab}(t') e^{-i(E_b - E_a)t'/\hbar} \left[ \int_0^{t'} H'_{ba}(t'') e^{i(E_b - E_a)t''/\hbar} dt'' \right] dt'$$

$$c_b^{(2)}(t) = b - \frac{ia}{\hbar} \int_0^t H'_{ba}(t') e^{i(E_b - E_a)t'/\hbar} dt' - \quad (6)$$

$$\frac{b}{\hbar^2} \int_0^t H'_{ba}(t') e^{i(E_b - E_a)t'/\hbar} \left[ \int_0^{t'} H'_{ab}(t'') e^{-i(E_b - E_a)t''/\hbar} dt'' \right] dt'$$

Since  $H'$  is independent of time, it can come outside the integral and since  $(H'_{ab})^* = H'_{ba}$  we have (where  $c_a(0) = a$  and  $c_b(0) = b$ ):

$$c_a^{(2)}(t) = a - \frac{ib}{\hbar} H'_{ab} \int_0^t e^{-i(E_b - E_a)t'/\hbar} dt' - \quad (7)$$

$$\begin{aligned} & \frac{a}{\hbar^2} |H'_{ab}|^2 \int_0^t \int_0^{t'} e^{-i(E_b - E_a)(t'' - t')/\hbar} dt'' dt' \\ & = a + \frac{bH'_{ab}}{E_b - E_a} \left( e^{-i(E_b - E_a)t/\hbar} - 1 \right) + \quad (8) \end{aligned}$$

$$\frac{a |H'_{ab}|^2}{(E_b - E_a)^2} \left( e^{-i(E_b - E_a)t/\hbar} + \frac{i}{\hbar} (E_b - E_a)t - 1 \right)$$

$$c_b^{(2)}(t) = b - \frac{ia}{\hbar} H'_{ba} \int_0^t e^{i(E_b - E_a)t'/\hbar} dt' - \quad (9)$$

$$\begin{aligned} & \frac{b}{\hbar^2} |H'_{ab}|^2 \int_0^t \int_0^{t'} e^{-i(E_b - E_a)(t'' - t')/\hbar} dt'' dt' \\ & = b - \frac{aH'_{ba}}{E_b - E_a} \left( e^{i(E_b - E_a)t/\hbar} - 1 \right) - \quad (10) \end{aligned}$$

$$\frac{b |H'_{ab}|^2}{(E_b - E_a)^2} \left( e^{i(E_b - E_a)t/\hbar} - \frac{i}{\hbar} (E_b - E_a)t - 1 \right)$$

To compare these results with the exact solution we worked out earlier, we set the initial conditions  $a = 1$  and  $b = 0$ , and define  $\omega_0 \equiv (E_b - E_a)/\hbar$ . Then

$$c_a^{(2)}(t) = 1 + \frac{|H'_{ab}|^2}{\hbar^2 \omega_0^2} (e^{-i\omega_0 t} + i\omega_0 t - 1) \quad (11)$$

$$c_b^{(2)}(t) = -\frac{H'_{ba}}{\hbar \omega_0} (e^{i\omega_0 t} - 1) \quad (12)$$

Our earlier exact formulas are

$$c_a(t) = e^{-i\omega_0 t/2} \left[ \cos(Qt) + \frac{i\omega_0}{2Q} \sin(Qt) \right] \quad (13)$$

$$c_b(t) = -\frac{i |H'_{ab}|}{\hbar Q} e^{i\omega_0 t/2} \sin(Qt) \quad (14)$$

where

$$Q \equiv \frac{1}{2} \sqrt{\omega_0^2 + \frac{4|H'_{ab}|^2}{\hbar^2}} \quad (15)$$

To compare these two sets of formulas, we need to Taylor expand the latter pair up to second order in  $H'$ . This is a messy procedure and is best consigned to Maple, from which we get

$$c_a(t) = 1 + \frac{|H'_{ab}|^2}{\hbar^2 \omega_0^2} (e^{-i\omega_0 t} + i\omega_0 t - 1) + \mathcal{O}(|H'|^4) \quad (16)$$

$$c_b(t) = -\frac{2i|H'_{ab}|}{\hbar\omega_0} e^{i\omega_0 t/2} \sin \frac{\omega_0 t}{2} + \mathcal{O}(|H'|^3) \quad (17)$$

$$= -\frac{2i|H'_{ab}|}{\hbar\omega_0} e^{i\omega_0 t/2} \frac{1}{2i} (e^{i\omega_0 t/2} - e^{-i\omega_0 t/2}) + \mathcal{O}(|H'|^3) \quad (18)$$

$$= -\frac{H'_{ba}}{\hbar\omega_0} (e^{i\omega_0 t} - 1) + \mathcal{O}(|H'|^3) \quad (19)$$

Thus the formulas agree up to second order.