SINUSOIDAL PERTURBATIONS IN TIME

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 9.7.

Using time-dependent perturbation theory we can get estimates of the coefficients in the solution of the Schrödinger equation with a time-dependent potential:

$$\Psi(x,t)=c_a(t)\,\psi_a(x)\,e^{-iE_at/\hbar}+c_b(t)\,\psi_b(x)\,e^{-iE_bt/\hbar} \tag{1}$$

where the complete hamiltonian is $H=H^0+H'$, ψ_a and ψ_b are the two eigenstates of H^0 and the coefficients are solutions of the coupled ODEs

$$\dot{c}_a = -\frac{i}{\hbar}\left[c_aH'_{aa}+c_bH'_{ab}e^{-i(E_b-E_a)t/\hbar}\right] \tag{2}$$

$$\dot{c}_b = -\frac{i}{\hbar}\left[c_bH'_{bb}+c_aH'_{ba}e^{i(E_b-E_a)t/\hbar}\right] \tag{3}$$

where

$$H'_{ij}\equiv\left\langle\psi_i\right|H'\left|\psi_j\right\rangle \tag{4}$$

If the diagonal matrix elements are zero, then we get the simpler ODEs:

$$\dot{c}_a = -\frac{i}{\hbar}H'_{ab}e^{-i\omega_0t}c_b \tag{5}$$

$$\dot{c}_b = -\frac{i}{\hbar}H'_{ba}e^{i\omega_0t}c_a \tag{6}$$

where

$$\omega_0\equiv\frac{E_b-E_a}{\hbar} \tag{7}$$

If the perturbation has a sinusoidal dependence on time such as

$$H'(\mathbf{r},t)=V(\mathbf{r})\cos\omega t \tag{8}$$

then

$$H'_{ab} = V_{ab}\cos\omega t \tag{9}$$

$$V_{ab} \equiv \langle\psi_a|V(\mathbf{r})|\psi_b\rangle \tag{10}$$

If we stop after the first-order step in the iterative solution we have

$$c_a^{(1)}(t) = 1 - \frac{i}{\hbar}\int_0^t H'_{aa}(t')\,dt' \tag{11}$$

$$= 1 \tag{12}$$

$$c_b^{(1)}(t) = -\frac{i}{\hbar}\int_0^t H'_{ba}(t')\,e^{i\omega_0 t}dt' \tag{13}$$

$$= -\frac{i}{\hbar}V_{ba}\int_0^t \cos\left(\omega t'\right)e^{i\omega_0 t'}dt' \tag{14}$$

$$= -\frac{i}{2\hbar}V_{ba}\int_0^t \left[e^{i(\omega_0+\omega)t'} + e^{i(\omega_0-\omega)t'}\right]dt' \tag{15}$$

$$= -\frac{V_{ba}}{2\hbar}\left[\frac{e^{i(\omega_0+\omega)t}-1}{\omega_0+\omega} + \frac{e^{i(\omega_0-\omega)t}-1}{\omega_0-\omega}\right] \tag{16}$$

At this point, we can study only frequencies $\omega \approx \omega_0$ and Griffiths does this in his section 9.1.3. The idea is that the first term can be dropped as the second term is much larger, due to the $\omega_0 - \omega$ in the denominator.

A different approach was taken by the Polish-American physicist Isidor Isaac Rabi (1898 - 1988). He noticed that this approximation is mathematically the same as taking the original perturbation to be

$$H' = \frac{V}{2}e^{-i\omega t} \tag{17}$$

so that

$$H'_{ba} = \frac{V_{ba}}{2}e^{-i\omega t} \tag{18}$$

$$H'_{ab} = \frac{V_{ab}}{2}e^{i\omega t} \tag{19}$$

In this case, we can solve the original ODEs 5 and 6 exactly without the need for perturbation theory. The equations become

$$\dot{c}_a = -\frac{i}{2\hbar} V_{ab} e^{i(\omega-\omega_0)t} c_b \tag{20}$$

$$\dot{c}_b = -\frac{i}{2\hbar} V_{ba} e^{-i(\omega-\omega_0)t} c_a \tag{21}$$

With the initial conditions $c_a(0) = 1$ and $c_b(0) = 0$, these equations are formally the same as those we solved earlier with H'_{ab} replaced by $V_{ab}/2$ and ω_0 replaced by $-(\omega-\omega_0)$. We can therefore write down the solution as

$$c_a(t) = e^{-i(\omega-\omega_0)t/2} \left[\cos(\omega_r t) - \frac{i(\omega-\omega_0)}{2\omega_r} \sin(\omega_r t)\right] \tag{22}$$

$$c_b(t) = -\frac{i|V_{ab}|}{2\hbar\omega_r} e^{-i(\omega-\omega_0)t/2} \sin(\omega_r t) \tag{23}$$

where Q in the earlier problem now becomes the Rabi flopping frequency ω_r:

$$\omega_r = \frac{1}{2}\sqrt{(\omega-\omega_0)^2 + \frac{|V_{ab}|^2}{\hbar^2}} \tag{24}$$

Note that

$$|c_a|^2 + |c_b|^2 = \cos^2(\omega_r t) + \frac{1}{4\omega_r^2} \sin^2(\omega_r t) \left[(\omega-\omega_0)^2 + \frac{|V_{ab}|^2}{\hbar^2}\right] \tag{25}$$

$$= 1 \tag{26}$$

The transition probability (for a flip from state a to state b) is $|c_b|^2$:

$$P_{a\to b} = |c_b|^2 = \frac{|V_{ab}|^2}{4\hbar^2\omega_r^2} \sin^2(\omega_r t) \tag{27}$$

The maximum probability occurs when $t = \pi/2\omega_r$ which gives

$$P_{a\to b} \le \frac{|V_{ab}|^2}{4\hbar^2\omega_r^2} \tag{28}$$

$$= \left[\frac{\hbar^2(\omega-\omega_0)^2}{|V_{ab}|^2} + 1\right]^{-1} < 1 \tag{29}$$

For small perturbations, $|V_{ab}| \ll \hbar|(\omega-\omega_0)|$ and if we expand 27 in a Taylor series about $V_{ab} = 0$ we get

$$\frac{|V_{ab}|^2}{4\hbar^2\omega_r^2}\sin^2(\omega_r t) = \frac{|V_{ab}|^2}{4\hbar^2}\left[\frac{4}{(\omega-\omega_0)^2}\sin^2\frac{(\omega_0-\omega)t}{2}+\ldots\right] \tag{30}$$

$$\approx \frac{|V_{ab}|^2}{\hbar^2(\omega-\omega_0)^2}\sin^2\frac{(\omega_0-\omega)t}{2} \tag{31}$$

which is the same as equation 9.28 in Griffiths.

The probability of the system being in its original state a is 1 at $t=0$ and next at $t=\pi/\omega_r$.

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