

SINUSOIDAL PERTURBATIONS IN TIME

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 9.7.

Using time-dependent perturbation theory we can get estimates of the coefficients in the solution of the Schrödinger equation with a time-dependent potential:

$$(0.1) \quad \Psi(x,t) = c_a(t) \psi_a(x) e^{-iE_a t/\hbar} + c_b(t) \psi_b(x) e^{-iE_b t/\hbar}$$

where the complete hamiltonian is $H = H^0 + H'$, ψ_a and ψ_b are the two eigenstates of H^0 and the coefficients are solutions of the coupled ODEs

$$(0.2) \quad \dot{c}_a = -\frac{i}{\hbar} \left[c_a H'_{aa} + c_b H'_{ab} e^{-i(E_b - E_a)t/\hbar} \right]$$

$$(0.3) \quad \dot{c}_b = -\frac{i}{\hbar} \left[c_b H'_{bb} + c_a H'_{ba} e^{i(E_b - E_a)t/\hbar} \right]$$

where

$$(0.4) \quad H'_{ij} \equiv \langle \psi_i | H' | \psi_j \rangle$$

If the diagonal matrix elements are zero, then we get the simpler ODEs:

$$(0.5) \quad \dot{c}_a = -\frac{i}{\hbar} H'_{ab} e^{-i\omega_0 t} c_b$$

$$(0.6) \quad \dot{c}_b = -\frac{i}{\hbar} H'_{ba} e^{i\omega_0 t} c_a$$

where

$$(0.7) \quad \omega_0 \equiv \frac{E_b - E_a}{\hbar}$$

If the perturbation has a sinusoidal dependence on time such as

$$(0.8) \quad H'(\mathbf{r}, t) = V(\mathbf{r}) \cos \omega t$$

then

$$(0.9) \quad H'_{ab} = V_{ab} \cos \omega t$$

$$(0.10) \quad V_{ab} \equiv \langle \psi_a | V(\mathbf{r}) | \psi_b \rangle$$

If we stop after the first-order step in the iterative solution we have

$$(0.11) \quad c_a^{(1)}(t) = 1 - \frac{i}{\hbar} \int_0^t H'_{aa}(t') dt'$$

$$(0.12) \quad = 1$$

$$(0.13) \quad c_b^{(1)}(t) = -\frac{i}{\hbar} \int_0^t H'_{ba}(t') e^{i\omega_0 t'} dt'$$

$$(0.14) \quad = -\frac{i}{\hbar} V_{ba} \int_0^t \cos(\omega t') e^{i\omega_0 t'} dt'$$

$$(0.15) \quad = -\frac{i}{2\hbar} V_{ba} \int_0^t \left[e^{i(\omega_0 + \omega)t'} + e^{i(\omega_0 - \omega)t'} \right] dt'$$

$$(0.16) \quad = -\frac{V_{ba}}{2\hbar} \left[\frac{e^{i(\omega_0 + \omega)t} - 1}{\omega_0 + \omega} + \frac{e^{i(\omega_0 - \omega)t} - 1}{\omega_0 - \omega} \right]$$

At this point, we can study only frequencies $\omega \approx \omega_0$ and Griffiths does this in his section 9.1.3. The idea is that the first term can be dropped as the second term is much larger, due to the $\omega_0 - \omega$ in the denominator.

A different approach was taken by the Polish-American physicist Isidor Isaac Rabi (1898 - 1988). He noticed that this approximation is mathematically the same as taking the original perturbation to be

$$(0.17) \quad H' = \frac{V}{2} e^{-i\omega t}$$

so that

$$(0.18) \quad H'_{ba} = \frac{V_{ba}}{2} e^{-i\omega t}$$

$$(0.19) \quad H'_{ab} = \frac{V_{ab}}{2} e^{i\omega t}$$

In this case, we can solve the original ODEs 0.5 and 0.6 exactly without the need for perturbation theory. The equations become

$$(0.20) \quad \dot{c}_a = -\frac{i}{2\hbar} V_{ab} e^{i(\omega - \omega_0)t} c_b$$

$$(0.21) \quad \dot{c}_b = -\frac{i}{2\hbar} V_{ba} e^{-i(\omega - \omega_0)t} c_a$$

With the initial conditions $c_a(0) = 1$ and $c_b(0) = 0$, these equations are formally the same as those we solved earlier with H'_{ab} replaced by $V_{ab}/2$ and ω_0 replaced by $-(\omega - \omega_0)$. We can therefore write down the solution as

$$(0.22) \quad c_a(t) = e^{-i(\omega - \omega_0)t/2} \left[\cos(\omega_r t) - \frac{i(\omega - \omega_0)}{2\omega_r} \sin(\omega_r t) \right]$$

$$(0.23) \quad c_b(t) = -\frac{i|V_{ab}|}{2\hbar\omega_r} e^{-i(\omega - \omega_0)t/2} \sin(\omega_r t)$$

where Q in the earlier problem now becomes the Rabi flopping frequency ω_r :

$$(0.24) \quad \omega_r = \frac{1}{2} \sqrt{(\omega - \omega_0)^2 + \frac{|V_{ab}|^2}{\hbar^2}}$$

Note that

$$(0.25) \quad |c_a|^2 + |c_b|^2 = \cos^2(\omega_r t) + \frac{1}{4\omega_r^2} \sin^2(\omega_r t) \left[(\omega - \omega_0)^2 + \frac{|V_{ab}|^2}{\hbar^2} \right]$$

$$(0.26) \quad = 1$$

The transition probability (for a flip from state a to state b) is $|c_b|^2$:

$$(0.27) \quad P_{a \rightarrow b} = |c_b|^2 = \frac{|V_{ab}|^2}{4\hbar^2 \omega_r^2} \sin^2(\omega_r t)$$

The maximum probability occurs when $t = \pi/2\omega_r$ which gives

$$(0.28) \quad P_{a \rightarrow b} \leq \frac{|V_{ab}|^2}{4\hbar^2 \omega_r^2}$$

$$(0.29) \quad = \left[\frac{\hbar^2 (\omega - \omega_0)^2}{|V_{ab}|^2} + 1 \right]^{-1} < 1$$

For small perturbations, $|V_{ab}| \ll \hbar |(\omega - \omega_0)|$ and if we expand 0.27 in a Taylor series about $V_{ab} = 0$ we get

$$(0.30) \quad \frac{|V_{ab}|^2}{4\hbar^2 \omega_r^2} \sin^2(\omega_r t) = \frac{|V_{ab}|^2}{4\hbar^2} \left[\frac{4}{(\omega - \omega_0)^2} \sin^2 \frac{(\omega_0 - \omega)t}{2} + \dots \right]$$

$$(0.31) \quad \approx \frac{|V_{ab}|^2}{\hbar^2 (\omega - \omega_0)^2} \sin^2 \frac{(\omega_0 - \omega)t}{2}$$

which is the same as equation 9.28 in Griffiths.

The probability of the system being in its original state a is 1 at $t = 0$ and next at $t = \pi/\omega_r$.

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