

SINUSOIDAL PERTURBATIONS IN TIME

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 9.7.

Using time-dependent perturbation theory we can get estimates of the coefficients in the solution of the Schrödinger equation with a time-dependent potential:

$$\Psi(x, t) = c_a(t) \psi_a(x) e^{-iE_a t/\hbar} + c_b(t) \psi_b(x) e^{-iE_b t/\hbar} \quad (1)$$

where the complete hamiltonian is $H = H^0 + H'$, ψ_a and ψ_b are the two eigenstates of H^0 and the coefficients are solutions of the coupled ODEs

$$\dot{c}_a = -\frac{i}{\hbar} \left[c_a H'_{aa} + c_b H'_{ab} e^{-i(E_b - E_a)t/\hbar} \right] \quad (2)$$

$$\dot{c}_b = -\frac{i}{\hbar} \left[c_b H'_{bb} + c_a H'_{ba} e^{i(E_b - E_a)t/\hbar} \right] \quad (3)$$

where

$$H'_{ij} \equiv \langle \psi_i | H' | \psi_j \rangle \quad (4)$$

If the diagonal matrix elements are zero, then we get the simpler ODEs:

$$\dot{c}_a = -\frac{i}{\hbar} H'_{ab} e^{-i\omega_0 t} c_b \quad (5)$$

$$\dot{c}_b = -\frac{i}{\hbar} H'_{ba} e^{i\omega_0 t} c_a \quad (6)$$

where

$$\omega_0 \equiv \frac{E_b - E_a}{\hbar} \quad (7)$$

If the perturbation has a sinusoidal dependence on time such as

$$H'(\mathbf{r}, t) = V(\mathbf{r}) \cos \omega t \quad (8)$$

then

$$H'_{ab} = V_{ab} \cos \omega t \quad (9)$$

$$V_{ab} \equiv \langle \psi_a | V(\mathbf{r}) | \psi_b \rangle \quad (10)$$

If we stop after the first-order step in the iterative solution we have

$$c_a^{(1)}(t) = 1 - \frac{i}{\hbar} \int_0^t H'_{aa}(t') dt' \quad (11)$$

$$= 1 \quad (12)$$

$$c_b^{(1)}(t) = -\frac{i}{\hbar} \int_0^t H'_{ba}(t') e^{i\omega_0 t'} dt' \quad (13)$$

$$= -\frac{i}{\hbar} V_{ba} \int_0^t \cos(\omega t') e^{i\omega_0 t'} dt' \quad (14)$$

$$= -\frac{i}{2\hbar} V_{ba} \int_0^t \left[e^{i(\omega_0 + \omega)t'} + e^{i(\omega_0 - \omega)t'} \right] dt' \quad (15)$$

$$= -\frac{V_{ba}}{2\hbar} \left[\frac{e^{i(\omega_0 + \omega)t} - 1}{\omega_0 + \omega} + \frac{e^{i(\omega_0 - \omega)t} - 1}{\omega_0 - \omega} \right] \quad (16)$$

At this point, we can study only frequencies $\omega \approx \omega_0$ and Griffiths does this in his section 9.1.3. The idea is that the first term can be dropped as the second term is much larger, due to the $\omega_0 - \omega$ in the denominator.

A different approach was taken by the Polish-American physicist Isidor Isaac Rabi (1898 - 1988). He noticed that this approximation is mathematically the same as taking the original perturbation to be

$$H' = \frac{V}{2} e^{-i\omega t} \quad (17)$$

so that

$$H'_{ba} = \frac{V_{ba}}{2} e^{-i\omega t} \quad (18)$$

$$H'_{ab} = \frac{V_{ab}}{2} e^{i\omega t} \quad (19)$$

In this case, we can solve the original ODEs 5 and 6 exactly without the need for perturbation theory. The equations become

$$\dot{c}_a = -\frac{i}{2\hbar} V_{ab} e^{i(\omega-\omega_0)t} c_b \quad (20)$$

$$\dot{c}_b = -\frac{i}{2\hbar} V_{ba} e^{-i(\omega-\omega_0)t} c_a \quad (21)$$

With the initial conditions $c_a(0) = 1$ and $c_b(0) = 0$, these equations are formally the same as those we solved earlier with H'_{ab} replaced by $V_{ab}/2$ and ω_0 replaced by $-(\omega - \omega_0)$. We can therefore write down the solution as

$$c_a(t) = e^{-i(\omega-\omega_0)t/2} \left[\cos(\omega_r t) - \frac{i(\omega-\omega_0)}{2\omega_r} \sin(\omega_r t) \right] \quad (22)$$

$$c_b(t) = -\frac{i|V_{ab}|}{2\hbar\omega_r} e^{-i(\omega-\omega_0)t/2} \sin(\omega_r t) \quad (23)$$

where Q in the earlier problem now becomes the Rabi flopping frequency ω_r :

$$\omega_r = \frac{1}{2} \sqrt{(\omega - \omega_0)^2 + \frac{|V_{ab}|^2}{\hbar^2}} \quad (24)$$

Note that

$$|c_a|^2 + |c_b|^2 = \cos^2(\omega_r t) + \frac{1}{4\omega_r^2} \sin^2(\omega_r t) \left[(\omega - \omega_0)^2 + \frac{|V_{ab}|^2}{\hbar^2} \right] \quad (25)$$

$$= 1 \quad (26)$$

The transition probability (for a flip from state a to state b) is $|c_b|^2$:

$$P_{a \rightarrow b} = |c_b|^2 = \frac{|V_{ab}|^2}{4\hbar^2\omega_r^2} \sin^2(\omega_r t) \quad (27)$$

The maximum probability occurs when $t = \pi/2\omega_r$ which gives

$$P_{a \rightarrow b} \leq \frac{|V_{ab}|^2}{4\hbar^2\omega_r^2} \quad (28)$$

$$= \left[\frac{\hbar^2(\omega - \omega_0)^2}{|V_{ab}|^2} + 1 \right]^{-1} < 1 \quad (29)$$

For small perturbations, $|V_{ab}| \ll \hbar|\omega - \omega_0|$ and if we expand 27 in a Taylor series about $V_{ab} = 0$ we get

$$\frac{|V_{ab}|^2}{4\hbar^2\omega_r^2} \sin^2(\omega_r t) = \frac{|V_{ab}|^2}{4\hbar^2} \left[\frac{4}{(\omega - \omega_0)^2} \sin^2 \frac{(\omega_0 - \omega)t}{2} + \dots \right] \quad (30)$$

$$\approx \frac{|V_{ab}|^2}{\hbar^2(\omega - \omega_0)^2} \sin^2 \frac{(\omega_0 - \omega)t}{2} \quad (31)$$

which is the same as equation 9.28 in Griffiths.

The probability of the system being in its original state a is 1 at $t = 0$ and next at $t = \pi/\omega_r$.

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