

TIME-DEPENDENT PERTURBATION THEORY: TRANSITION PROBABILITIES IN A MULTI-STATE SYSTEM

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 9.16.

In time-dependent perturbation theory for a multi-state system we got a general expression for the coefficient $c_m(t)$ to first order. If the system starts in state N then for $m \neq N$:

$$(1) \quad c_m(t) = -\frac{i}{\hbar} \int_0^t H'_{mN} e^{i(E_m - E_N)t'/\hbar} dt'$$

and

$$(2) \quad c_N^{(1)} = 1 - \frac{i}{\hbar} \int_0^t H'_{NN}(t') dt'$$

We looked at two examples of a perturbation: when the perturbation is constant between the times $t = 0$ and t , and where the perturbation is sinusoidal in the same interval. For the constant perturbation, we get

$$(3) \quad c_{m \neq N}^{(1)}(t) = -\frac{2iH'_{mN}}{(E_m - E_N)} e^{i(E_m - E_N)t/2\hbar} \sin \frac{(E_m - E_N)t}{2\hbar}$$

$$(4) \quad c_N^{(1)}(t) = 1 - \frac{i}{\hbar} H'_{NN}t$$

Since H' is hermitian, the matrix elements H'_{ij} are real, so the sum of all the transition probabilities is (we'll drop the superscript (1) from now on, as we're always dealing with the first order coefficients):

$$(5) \quad \sum_m |c_m(t)|^2 = 1 + \frac{|H'_{NN}|^2 t^2}{\hbar^2} + \sum_{m \neq N} |c_m(t)|^2$$

$$(6) \quad = 1 + \frac{|H'_{NN}|^2 t^2}{\hbar^2} + 4 \sum_{m \neq N} \frac{|H'_{mN}|^2}{(E_m - E_N)^2} \sin^2 \frac{(E_m - E_N)t}{2\hbar}$$

This is clearly greater than 1, but all the extra terms are second order in the matrix elements of the perturbation H'_{ij} , so to first order, we're safe.

For the sinusoidal perturbation $H'(t) = V \cos(\omega t)$, we got

$$(7) \quad c_{m \neq N}(t) = -V_{mN} i \frac{e^{i(E_m - E_N \pm \hbar\omega)t/2\hbar}}{E_N - E_m \pm \hbar\omega} \sin \frac{(E_N - E_m \pm \hbar\omega)t}{2\hbar}$$

$$(8) \quad c_N(t) = 1 - \frac{i}{\hbar} V_{NN} \int_0^t \cos(\omega t') dt'$$

$$(9) \quad = 1 - \frac{iV_{NN}}{\hbar\omega} \sin \omega t$$

The sum of all the transition probabilities is

$$(10)$$

$$\sum_m |c_m(t)|^2 = 1 + \frac{|V_{NN}|^2}{\hbar^2 \omega^2} \sin^2 \omega t + \sum_{m \neq N} |c_m(t)|^2$$

$$(11)$$

$$= 1 + \frac{|V_{NN}|^2}{\hbar^2 \omega^2} \sin^2 \omega t + \sum_{m \neq N} \frac{|V_{mN}|^2}{(E_N - E_m \pm \hbar\omega)^2} \sin^2 \frac{(E_N - E_m \pm \hbar\omega)t}{2\hbar}$$

Again, the sum is greater than 1 but all the offending terms are of second order.

To calculate the probability of the system remaining in state N it would seem better to use $1 - \sum_{m \neq N} |c_m(t)|^2$ rather than $|c_N(t)|^2$ since the latter is greater than 1.