STIMULATED EMISSION OF RADIATION: LASERS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Section 9.2; Problem 9.19.

A particularly important case that we can analyze using time-dependent perturbation theory is that of the perturbation of an atom by an electromagnetic wave. A proper quantum treatment of the problem requires, of course, a proper quantum theory of electromagnetism (that is, quantum electrodynamics) but we haven't got there yet, so we will follow Griffiths and use classical electromagnetism in combination with quantum mechanics.

We'll look at what happens if a monochromatic plane wave encounters an electron in an atom, where the electron can be in one of two states (for example, the ground state and first excited state). To start, we'll assume that the wave is travelling in the $+y$ direction and is polarized in the z direction (that is, the electric field points in the $\pm z$ direction, which is perpendicular to the magnetic field). We can write the electric field as

$$\mathbf{E}=E_0\hat{\mathbf{z}}\cos\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right) \tag{0.1}$$

where $\mathbf{k}$ is the direction of propagation and $\omega=kc$ is the angular frequency. If we assume that the wavelength of the light is much larger than the atom (true for visible light and longer wavelengths), then we can drop the $\mathbf{k}\cdot\mathbf{r}$ term since it's effectively constant over the region of the atom, and we get

$$\mathbf{E}=E_0\hat{\mathbf{z}}\cos\left(\omega t\right) \tag{0.2}$$

To get the perturbation H' to the hamiltonian, we can use the formula for the energy of a point charge in an electric field. The work done in moving a charge q from point $\mathbf{a}$ to point $\mathbf{b}$ is

$$W=-q\int_{\mathbf{a}}^{\mathbf{b}}\mathbf{E}\cdot d\mathbf{l} \tag{0.3}$$

If we take $\mathbf{a}=0$ to be the centre of the atom and $\mathbf{b}$ to be the field point $\mathbf{r}$ then since $\mathbf{E}$ points in the z direction and, at a given time, is constant over the region of the atom, we get

$$W=-qE_0\hat{\mathbf{z}}\cdot\mathbf{r}\cos(\omega t)=-qE_0z\cos(\omega t) \tag{0.4}$$

The perturbation is therefore

$$H'=-qE_0z\cos(\omega t) \tag{0.5}$$

and the matrix elements are

$$H'_{ba} = -\langle\psi_b|z|\psi_a\rangle qE_0\cos\omega t \tag{0.6}$$
$$\equiv -\mathfrak{p}E_0\cos\omega t \tag{0.7}$$
$$\mathfrak{p} \equiv q\langle\psi_b|z|\psi_a\rangle \tag{0.8}$$

The quantity $\mathfrak{p}$ is effectively a dipole moment, as the bracket factor is the average displacement of the electron from the origin. Assuming that the diagonal elements of H' are zero (as is true for hydrogen) we have a system with sinusoidal perturbations in time that we treated earlier, with $V_{ab}=-\mathfrak{p}E_0$, so we can write down the solution. (Actually, we'll use the perturbation theory result - equation 9.28 in Griffiths - rather than Rabi's more accurate result).

$$P_{a\to b}(t)=\frac{|V_{ab}|^2}{\hbar^2(\omega-\omega_0)^2}\sin^2\frac{(\omega_0-\omega)t}{2} \tag{0.9}$$

$$=\left(\frac{|\mathfrak{p}|E_0}{\hbar}\right)^2\frac{\sin^2\frac{(\omega_0-\omega)t}{2}}{(\omega-\omega_0)^2} \tag{0.10}$$

where $\omega_0=(E_b-E_a)/\hbar$ is the frequency corresponding to the difference in energy levels between states a and b. That is, the probability of a transition from state a (the ground state) to state b (the excited state) oscillates in time. This is the phenomenon of *stimulated absorption*, in which the electric field causes the electron to jump to a higher energy level. This actually happens by absorbing a photon, although using a non-quantum theory of electrodynamics, we can't predict that.

Curiously, if we analyze the situation where the electron starts out in state b, we get the same result. To see this, look back at the derivation of $P_{a\to b}$ that we did earlier. If we start with initial conditions $c_a(0)=0$ and $c_b(0)=1$ so that the electron starts in state b, we can just interchange a and b all the way through the derivation. This results in ω_0 being replaced by $-\omega_0=(E_a-E_b)/\hbar$ so we get

$$c_a(t) = -\frac{V_{ab}}{2\hbar}\left[\frac{e^{i(-\omega_0+\omega)t}-1}{-\omega_0+\omega} - \frac{e^{i(-\omega_0-\omega)t}-1}{\omega_0+\omega}\right] \tag{0.11}$$

In the approximation, we now keep only the first term (since $\omega \approx \omega_0$) so we now get

$$c_a(t) = -\frac{iV_{ab}}{\hbar(\omega-\omega_0)} e^{i(\omega-\omega_0)t/2} \sin\frac{(\omega-\omega_0)t}{2} \tag{0.12}$$

$$P_{b\to a}(t) = |c_a(t)|^2 = \frac{|V_{ab}|^2}{\hbar^2(\omega-\omega_0)^2} \sin^2\frac{(\omega-\omega_0)t}{2} \tag{0.13}$$

In our case this gives

$$P_{b\to a}(t) = |\langle\psi_a|z|\psi_b\rangle|^2 \left(\frac{E_0}{\hbar}\right)^2 \frac{\sin^2\frac{(\omega_0-\omega)t}{2}}{(\omega-\omega_0)^2} \tag{0.14}$$

The first term is

$$|\langle\psi_a|z|\psi_b\rangle|^2 = \left|\langle\psi_b|z|\psi_a\rangle^*\right|^2 \tag{0.15}$$

$$= |\langle\psi_b|z|\psi_a\rangle|^2 \tag{0.16}$$

$$= |\mathfrak{p}|^2 \tag{0.17}$$

so

$$P_{b\to a}(t) = P_{a\to b}(t) \tag{0.18}$$

That is, it is just as likely for an electromagnetic wave to stimulate the emission of a photon as it is to stimulate the absorption of one. This has a profound application in modern technology: the laser (which is an acronym for Light Amplification by Stimulated Emission of Radiation). If we can arrange to have a collection of atoms initially in the excited state b, then a small input of electromagnetic radiation will stimulate a few atoms to emit photons. These photons, in turn, will stimulate other atoms to emit photons giving a cascading effect. In other words, the initial light pulse is amplified by stimulated emission.

As we'll see, in addition to stimulated absorption and emission, there is also the phenomenon of *spontaneous emission*, in which an atom in an excited state decays to a lower state and emits a photon without the intervention of an external field. Actually, it turns out that spontaneous emission is due to the interaction of an excited atom with a 'zero-point' electrodynamic field, which is something that arises only in quantum electrodynamics.

By the way, there is no such thing as 'spontaneous absorption' (absorption without an external field), since this would essentially require the acquisition of energy (a photon) from nothing.

PINGBACKS

Pingback: Spontaneous emission: Einstein's argument
Pingback: Spontaneous emission from the zero point field
Pingback: Selection rules for spontaneous emission of radiation
Pingback: Time-dependent perturbation theory for a multi-level system
Pingback: Stimulated emission of radiation at high frequencies