

ELECTRON IN A PRECESSING MAGNETIC FIELD

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 10.2.

For a spin 1/2 particle in a magnetic field \mathbf{B} , we've seen that the hamiltonian is

$$H = -\gamma \mathbf{B} \cdot \mathbf{S} \quad (1)$$

where γ is the gyromagnetic ratio, which for an electron is $-e/m$. Now suppose that the magnetic field's direction precesses around the z axis (sweeps out a cone) with angular speed ω , so that \mathbf{B} makes an angle α with the z axis. That is

$$\mathbf{B}(t) = B_0 [\sin \alpha \cos(\omega t) \hat{\mathbf{x}} + \sin \alpha \sin(\omega t) \hat{\mathbf{y}} + \cos \alpha \hat{\mathbf{z}}] \quad (2)$$

At time t , the component of \mathbf{S} along \mathbf{B} is given by

$$S_B = \frac{\hbar}{2} \begin{pmatrix} \cos \alpha & \sin \alpha e^{-i\omega t} \\ \sin \alpha e^{i\omega t} & -\cos \alpha \end{pmatrix} \quad (3)$$

so the hamiltonian is

$$H = \frac{\hbar \omega_1}{2} \begin{pmatrix} \cos \alpha & \sin \alpha e^{-i\omega t} \\ \sin \alpha e^{i\omega t} & -\cos \alpha \end{pmatrix} \quad (4)$$

$$\omega_1 \equiv \frac{eB_0}{m} \quad (5)$$

If we freeze the system at time t and solve the time-independent Schrödinger equation to get the eigenvalues and eigenspinors we get

$$\chi_+ = \begin{pmatrix} \cos(\alpha/2) \\ e^{i\omega t} \sin(\alpha/2) \end{pmatrix} \quad (6)$$

$$\chi_- = \begin{pmatrix} e^{-i\omega t} \sin(\alpha/2) \\ -\cos(\alpha/2) \end{pmatrix} \quad (7)$$

with energies

$$E_{\pm} = \pm \frac{\hbar\omega_1}{2} \quad (8)$$

Griffiths gives the exact solution to the time-dependent Schrödinger equation for this problem as

$$\chi(t) = \begin{bmatrix} \left(\cos \frac{\lambda t}{2} - i \frac{\omega_1 - \omega}{\lambda} \sin \frac{\lambda t}{2} \right) \cos \frac{\alpha}{2} e^{-i\omega t/2} \\ \left(\cos \frac{\lambda t}{2} - i \frac{\omega_1 + \omega}{\lambda} \sin \frac{\lambda t}{2} \right) \sin \frac{\alpha}{2} e^{i\omega t/2} \end{bmatrix} \quad (9)$$

where

$$\lambda \equiv \sqrt{\omega^2 + \omega_1^2 - 2\omega\omega_1 \cos \alpha} \quad (10)$$

To prove this, we need to show that

$$H\chi = i\hbar \frac{\partial \chi}{\partial t} \quad (11)$$

As usual, I'll use Maple to help things along, although even Maple requires a bit of help here and there. We'll start with $H\chi$ which is the matrix product of 4 and 9. After multiplying out the terms and using the trig identities $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$ we get

$$H\chi = \frac{\hbar\omega_1}{2\lambda} \begin{bmatrix} e^{-i\omega t/2} \cos \frac{\alpha}{2} \left[i \left(4\omega \cos^2 \frac{\alpha}{2} - 3\omega - \omega_1 \right) \sin \frac{\lambda t}{2} + \lambda \cos \frac{\lambda t}{2} \right] \\ e^{i\omega t/2} \sin \frac{\alpha}{2} \left[i \left(4\omega \cos^2 \frac{\alpha}{2} - \omega - \omega_1 \right) \sin \frac{\lambda t}{2} + \lambda \cos \frac{\lambda t}{2} \right] \end{bmatrix} \quad (12)$$

Now for the RHS. We get after collecting terms

$$i\hbar \frac{\partial \chi}{\partial t} = \frac{\hbar}{2\lambda} \begin{bmatrix} e^{-i\omega t/2} \cos \frac{\alpha}{2} \left[i \left(\omega^2 - \omega\omega_1 - \lambda^2 \right) \sin \frac{\lambda t}{2} + \lambda\omega_1 \cos \frac{\lambda t}{2} \right] \\ e^{i\omega t/2} \sin \frac{\alpha}{2} \left[i \left(\omega^2 + \omega\omega_1 - \lambda^2 \right) \sin \frac{\lambda t}{2} + \lambda\omega_1 \cos \frac{\lambda t}{2} \right] \end{bmatrix} \quad (13)$$

The two sides are equal if both the following are true:

$$\left(4\omega \cos^2 \frac{\alpha}{2} - 3\omega - \omega_1 \right) \omega_1 = \omega^2 - \omega\omega_1 - \lambda^2 \quad (14)$$

$$\left(4\omega \cos^2 \frac{\alpha}{2} - \omega - \omega_1 \right) \omega_1 = \omega^2 + \omega\omega_1 - \lambda^2 \quad (15)$$

Substituting from 10 both equations give the same condition:

$$4\omega\omega_1 \cos^2 \frac{\alpha}{2} = 2\omega\omega_1 (1 + \cos \alpha) \quad (16)$$

$$\cos \alpha = 2 \cos^2 \frac{\alpha}{2} - 1 \quad (17)$$

The last line is a trig identity, so the time-dependent Schrödinger equation is satisfied.

We can also express 9 as a linear combination of 6 and 7. Griffiths gives the answer as

$$\begin{aligned} \chi(t) = & \left[\cos \frac{\lambda t}{2} - i \frac{(\omega_1 - \omega \cos \alpha)}{\lambda} \sin \frac{\lambda t}{2} \right] e^{-i\omega t/2} \chi_+(t) + \\ & i \left[\frac{\omega}{\lambda} \sin \alpha \sin \frac{\lambda t}{2} \right] e^{\omega t/2} \chi_-(t) \end{aligned} \quad (18)$$

This can be verified by direct calculation. Take the top element first and use the cosine of difference of angles formula:

$$\begin{aligned} \chi_1 = & \left[\cos \frac{\lambda t}{2} - i \frac{(\omega_1 - \omega \cos \alpha)}{\lambda} \sin \frac{\lambda t}{2} \right] e^{-i\omega t/2} \cos \frac{\alpha}{2} + \\ & i \left[\frac{\omega}{\lambda} \sin \alpha \sin \frac{\lambda t}{2} \right] e^{\omega t/2} e^{-i\omega t} \sin \frac{\alpha}{2} \end{aligned} \quad (19)$$

$$= e^{-i\omega t/2} \left[\cos \frac{\lambda t}{2} \cos \frac{\alpha}{2} + \frac{i}{\lambda} \sin \frac{\lambda t}{2} \left(-\omega_1 \cos \frac{\alpha}{2} + \omega \cos \frac{\alpha}{2} \right) \right] \quad (20)$$

$$= \left(\cos \frac{\lambda t}{2} - i \frac{\omega_1 - \omega}{\lambda} \sin \frac{\lambda t}{2} \right) \cos \frac{\alpha}{2} e^{-i\omega t/2} \quad (21)$$

The bottom element works much the same way, using the sine of difference of angles formula:

$$\begin{aligned} \chi_2 = & \left[\cos \frac{\lambda t}{2} - i \frac{(\omega_1 - \omega \cos \alpha)}{\lambda} \sin \frac{\lambda t}{2} \right] e^{-i\omega t/2} e^{i\omega t} \sin \frac{\alpha}{2} - \\ & i \left[\frac{\omega}{\lambda} \sin \alpha \sin \frac{\lambda t}{2} \right] e^{\omega t/2} \cos \frac{\alpha}{2} \end{aligned} \quad (22)$$

$$= \left(\cos \frac{\lambda t}{2} - i \frac{\omega_1 + \omega}{\lambda} \sin \frac{\lambda t}{2} \right) \sin \frac{\alpha}{2} e^{i\omega t/2} \quad (23)$$

Finally, writing $\chi(t) = c_+(t) \chi_+ + c_-(t) \chi_-$ we can check that the coefficients are normalized.

$$|c_+|^2 + |c_-|^2 = \cos^2 \frac{\lambda t}{2} + \frac{(\omega_1 - \omega \cos \alpha)^2}{\lambda^2} \sin^2 \frac{\lambda t}{2} + \left[\frac{\omega}{\lambda} \sin \alpha \sin \frac{\lambda t}{2} \right]^2 \quad (24)$$

$$= \cos^2 \frac{\lambda t}{2} + \sin^2 \frac{\lambda t}{2} \left[\frac{(\omega_1 - \omega \cos \alpha)^2}{\lambda^2} + \left(\frac{\omega}{\lambda} \sin \alpha \right)^2 \right] \quad (25)$$

$$= \cos^2 \frac{\lambda t}{2} + \sin^2 \frac{\lambda t}{2} \left[\frac{\omega^2 + \omega_1^2 - 2\omega\omega_1 \cos \alpha}{\lambda^2} \right] \quad (26)$$

$$= \cos^2 \frac{\lambda t}{2} + \sin^2 \frac{\lambda t}{2} \left[\frac{\omega^2 + \omega_1^2 - 2\omega\omega_1 \cos \alpha}{\omega^2 + \omega_1^2 - 2\omega\omega_1 \cos \alpha} \right] \quad (27)$$

$$= 1 \quad (28)$$

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