ELECTRON IN A PRECESSING MAGNETIC FIELD

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 10.2.

For a spin 1/2 particle in a magnetic field $\mathbf{B}$, we've seen that the hamiltonian is

$$\mathsf{H} = -\gamma \mathbf{B} \cdot \mathsf{S} \tag{1}$$

where γ is the gyromagnetic ratio, which for an electron is $-e/m$. Now suppose that the magnetic field's direction precesses around the z axis (sweeps out a cone) with angular speed ω, so that $\mathbf{B}$ makes an angle α with the z axis. That is

$$\mathbf{B}(t) = B_0 [\sin\alpha\cos(\omega t)\hat{\mathbf{x}} + \sin\alpha\sin(\omega t)\hat{\mathbf{y}} + \cos\alpha\hat{\mathbf{z}}] \tag{2}$$

At time t, the component of $\mathbf{S}$ along $\mathbf{B}$ is given by

$$\mathsf{S}_B = \frac{\hbar}{2}\begin{pmatrix} \cos\alpha & \sin\alpha e^{-i\omega t} \\ \sin\alpha e^{i\omega t} & -\cos\alpha \end{pmatrix} \tag{3}$$

so the hamiltonian is

$$\mathsf{H} = \frac{\hbar\omega_1}{2}\begin{pmatrix} \cos\alpha & \sin\alpha e^{-i\omega t} \\ \sin\alpha e^{i\omega t} & -\cos\alpha \end{pmatrix} \tag{4}$$

$$\omega_1 \equiv \frac{eB_0}{m} \tag{5}$$

If we freeze the system at time t and solve the time-independent Schrödinger equation to get the eigenvalues and eigenspinors we get

$$\chi_+ = \begin{pmatrix} \cos(\alpha/2) \\ e^{i\omega t}\sin(\alpha/2) \end{pmatrix} \tag{6}$$

$$\chi_- = \begin{pmatrix} e^{-i\omega t}\sin(\alpha/2) \\ -\cos(\alpha/2) \end{pmatrix} \tag{7}$$

with energies

$$E_{\pm}=\pm\frac{\hbar\omega_{1}}{2} \tag{8}$$

Griffiths gives the exact solution to the time-dependent Schrödinger equation for this problem as

$$\chi(t)=\left[\begin{array}{c}\left(\cos\frac{\lambda t}{2}-i\frac{\omega_{1}-\omega}{\lambda}\sin\frac{\lambda t}{2}\right)\cos\frac{\alpha}{2}e^{-i\omega t/2}\\ \left(\cos\frac{\lambda t}{2}-i\frac{\omega_{1}+\omega}{\lambda}\sin\frac{\lambda t}{2}\right)\sin\frac{\alpha}{2}e^{i\omega t/2}\end{array}\right] \tag{9}$$

where

$$\lambda\equiv\sqrt{\omega^{2}+\omega_{1}^{2}-2\omega\omega_{1}\cos\alpha} \tag{10}$$

To prove this, we need to show that

$$\mathsf{H}\chi=i\hbar\frac{\partial\chi}{\partial t} \tag{11}$$

As usual, I'll use Maple to help things along, although even Maple requires a bit of help here and there. We'll start with $\mathsf{H}\chi$ which is the matrix product of 4 and 9. After multiplying out the terms and using the trig identities $\cos(a\pm b)=\cos a\cos b\mp\sin a\sin b$ we get

$$\mathsf{H}\chi=\frac{\hbar\omega_{1}}{2\lambda}\left[\begin{array}{c}e^{-i\omega t/2}\cos\frac{\alpha}{2}\left[i\left(4\omega\cos^{2}\frac{\alpha}{2}-3\omega-\omega_{1}\right)\sin\frac{\lambda t}{2}+\lambda\cos\frac{\lambda t}{2}\right]\\ e^{i\omega t/2}\sin\frac{\alpha}{2}\left[i\left(4\omega\cos^{2}\frac{\alpha}{2}-\omega-\omega_{1}\right)\sin\frac{\lambda t}{2}+\lambda\cos\frac{\lambda t}{2}\right]\end{array}\right] \tag{12}$$

Now for the RHS. We get after collecting terms

$$i\hbar\frac{\partial\chi}{\partial t}=\frac{\hbar}{2\lambda}\left[\begin{array}{c}e^{-i\omega t/2}\cos\frac{\alpha}{2}\left[i\left(\omega^{2}-\omega\omega_{1}-\lambda^{2}\right)\sin\frac{\lambda t}{2}+\lambda\omega_{1}\cos\frac{\lambda t}{2}\right]\\ e^{i\omega t/2}\sin\frac{\alpha}{2}\left[i\left(\omega^{2}+\omega\omega_{1}-\lambda^{2}\right)\sin\frac{\lambda t}{2}+\lambda\omega_{1}\cos\frac{\lambda t}{2}\right]\end{array}\right] \tag{13}$$

The two sides are equal if both the following are true:

$$\left(4\omega\cos^{2}\frac{\alpha}{2}-3\omega-\omega_{1}\right)\omega_{1} = \omega^{2}-\omega\omega_{1}-\lambda^{2} \tag{14}$$

$$\left(4\omega\cos^{2}\frac{\alpha}{2}-\omega-\omega_{1}\right)\omega_{1} = \omega^{2}+\omega\omega_{1}-\lambda^{2} \tag{15}$$

Substituting from 10 both equations give the same condition:

$$4\omega\omega_1\cos^2\frac{\alpha}{2} = 2\omega\omega_1\left(1+\cos\alpha\right) \tag{16}$$

$$\cos\alpha = 2\cos^2\frac{\alpha}{2}-1 \tag{17}$$

The last line is a trig identity, so the time-dependent Schrödinger equation is satisfied.

We can also express 9 as a linear combination of 6 and 7. Griffiths gives the answer as

$$\begin{aligned}\chi\left(t\right) = \left[\cos\frac{\lambda t}{2}-i\frac{\left(\omega_1-\omega\cos\alpha\right)}{\lambda}\sin\frac{\lambda t}{2}\right]e^{-i\omega t/2}\chi_+\left(t\right)+ \\ i\left[\frac{\omega}{\lambda}\sin\alpha\sin\frac{\lambda t}{2}\right]e^{\omega t/2}\chi_-\left(t\right)\end{aligned} \tag{18}$$

This can be verified by direct calculation. Take the top element first and use the cosine of difference of angles formula:

$$\begin{aligned}\chi_1 &= \left[\cos\frac{\lambda t}{2}-i\frac{\left(\omega_1-\omega\cos\alpha\right)}{\lambda}\sin\frac{\lambda t}{2}\right]e^{-i\omega t/2}\cos\frac{\alpha}{2}+ \\ &\quad i\left[\frac{\omega}{\lambda}\sin\alpha\sin\frac{\lambda t}{2}\right]e^{\omega t/2}e^{-i\omega t}\sin\frac{\alpha}{2} && (19)\\ &= e^{-i\omega t/2}\left[\cos\frac{\lambda t}{2}\cos\frac{\alpha}{2}+\frac{i}{\lambda}\sin\frac{\lambda t}{2}\left(-\omega_1\cos\frac{\alpha}{2}+\omega\cos\frac{\alpha}{2}\right)\right] && (20)\\ &= \left(\cos\frac{\lambda t}{2}-i\frac{\omega_1-\omega}{\lambda}\sin\frac{\lambda t}{2}\right)\cos\frac{\alpha}{2}e^{-i\omega t/2} && (21)\end{aligned}$$

The bottom element works much the same way, using the sine of difference of angles formula:

$$\begin{aligned}\chi_2 &= \left[\cos\frac{\lambda t}{2}-i\frac{\left(\omega_1-\omega\cos\alpha\right)}{\lambda}\sin\frac{\lambda t}{2}\right]e^{-i\omega t/2}e^{i\omega t}\sin\frac{\alpha}{2}- \\ &\quad i\left[\frac{\omega}{\lambda}\sin\alpha\sin\frac{\lambda t}{2}\right]e^{\omega t/2}\cos\frac{\alpha}{2} && (22)\\ &= \left(\cos\frac{\lambda t}{2}-i\frac{\omega_1+\omega}{\lambda}\sin\frac{\lambda t}{2}\right)\sin\frac{\alpha}{2}e^{i\omega t/2} && (23)\end{aligned}$$

Finally, writing $\chi\left(t\right)=c_+\left(t\right)\chi_+ + c_-\left(t\right)\chi_-$ we can check that the coefficients are normalized.

$$|c_+|^2+|c_-|^2=\cos^2\frac{\lambda t}{2}+\frac{(\omega_1-\omega\cos\alpha)^2}{\lambda^2}\sin^2\frac{\lambda t}{2}+\left[\frac{\omega}{\lambda}\sin\alpha\sin\frac{\lambda t}{2}\right]^2 \tag{24}$$

$$=\cos^2\frac{\lambda t}{2}+\sin^2\frac{\lambda t}{2}\left[\frac{(\omega_1-\omega\cos\alpha)^2}{\lambda^2}+\left(\frac{\omega}{\lambda}\sin\alpha\right)^2\right] \tag{25}$$

$$=\cos^2\frac{\lambda t}{2}+\sin^2\frac{\lambda t}{2}\left[\frac{\omega^2+\omega_1^2-2\omega\omega_1\cos\alpha}{\lambda^2}\right] \tag{26}$$

$$=\cos^2\frac{\lambda t}{2}+\sin^2\frac{\lambda t}{2}\left[\frac{\omega^2+\omega_1^2-2\omega\omega_1\cos\alpha}{\omega^2+\omega_1^2-2\omega\omega_1\cos\alpha}\right] \tag{27}$$

$$=1 \tag{28}$$

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