

PHASES IN THE ADIABATIC APPROXIMATION

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 10.3.

The adiabatic theorem (see Griffiths, section 10.1 for a proof) says that if a system starts out in the n th state of a time-dependent hamiltonian, and the hamiltonian changes slowly compared to the internal period of the time-independent wave function (that is, the time scale over which the hamiltonian changes is much longer than \hbar/E_n), then after a time t the system will end up in state

$$(1) \quad \Psi_n(t) = e^{i\theta_n(t)} e^{i\gamma_n(t)} \psi_n(t)$$

where

$$(2) \quad \theta_n(t) \equiv -\frac{1}{\hbar} \int_0^t E_n(t') dt'$$

$$(3) \quad \gamma_n(t) \equiv i \int_0^t \left\langle \psi_n(t') \left| \frac{\partial}{\partial t'} \psi_n(t') \right. \right\rangle dt'$$

θ is called the *dynamic phase* and γ is called the *geometric phase*.

The wave functions $\psi_n(t)$ are the solutions of the eigenvalue equation at a particular time t :

$$(4) \quad H(t) \psi_n(t) = E_n(t) \psi_n(t)$$

That is, they aren't a full solution of the time dependent Schrödinger equation; rather they are the solutions of the time-independent Schrödinger equation with whatever parameters are now time-dependent in the hamiltonian replaced by their time-dependent forms.

For example, with an infinite square well whose right wall moves so that its position w is a function of time $w(t)$, we have

$$(5) \quad \psi_n(t) = \sqrt{\frac{2}{w(t)}} \sin \frac{n\pi}{w(t)} x$$

$$(6) \quad E_n(t) = \frac{(n\pi\hbar)^2}{2mw^2(t)}$$

In this case, ψ_n depends on only one time-dependent parameter, so we can use the chain rule to write

$$(7) \quad \gamma_n(t) = i \int_0^t \left\langle \psi_n \left| \frac{\partial}{\partial w} \psi_n \right. \right\rangle \frac{dw}{dt'} dt'$$

$$(8) \quad = i \int_{w_1}^{w_2} \left\langle \psi_n \left| \frac{\partial}{\partial w} \psi_n \right. \right\rangle dw$$

where the wall moves from w_1 to w_2 between times 0 and t . We get

$$(9) \quad \frac{\partial}{\partial w} \psi_n = -\frac{\sqrt{2}}{2w^{5/2}} \left[w \sin \frac{n\pi}{w} x + 2n\pi x \cos \frac{n\pi}{w} x \right]$$

$$(10) \quad \left\langle \psi_n \left| \frac{\partial}{\partial w} \psi_n \right. \right\rangle = -\frac{1}{w^3} \int_0^w \sin \frac{n\pi}{w} x \left[w \sin \frac{n\pi}{w} x + 2n\pi x \cos \frac{n\pi}{w} x \right] dx$$

$$(11) \quad = \frac{\sin^2 n\pi}{w}$$

$$(12) \quad = 0$$

In this case, there is no change in phase due to the geometric phase. In fact, we can see this is generally true for real wave functions ψ_n since

$$(13) \quad \langle \psi_n | \psi_n \rangle = 1$$

$$(14) \quad \frac{d}{dt} \langle \psi_n | \psi_n \rangle = 0$$

$$(15) \quad = \langle \dot{\psi}_n | \psi_n \rangle + \langle \psi_n | \dot{\psi}_n \rangle$$

$$(16) \quad = 2\Re(\langle \dot{\psi}_n | \psi_n \rangle)$$

That is, $\left\langle \psi_n(t') \left| \frac{\partial}{\partial t'} \psi_n(t') \right. \right\rangle$ must be purely imaginary, so if ψ_n is real, the bracket must be zero. This also means that γ is always real.

Thus γ is zero as the wall moves from w_1 to w_2 and also as it moves back from w_2 to w_1 .

The dynamic phase for the same journey is

$$(17) \quad \theta_n(t) = -\frac{1}{\hbar} \int_0^t E_n(t') dt'$$

$$(18) \quad = -\frac{\hbar(n\pi)^2}{2m} \int_0^t \frac{1}{w^2(t')} dt'$$

If the speed of the wall is constant so that $w = w_1 + vt$ we have

$$(19) \quad \theta_n(t) = -\frac{\hbar(n\pi)^2}{2m} \int_0^{(w_2-w_1)/v} \frac{dt'}{(w_1 + vt')^2}$$

$$(20) \quad = \frac{\hbar(n\pi)^2}{2mv} \frac{w_1 - w_2}{w_1 w_2}$$

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