

## BERRY'S PHASE: DEFINITION AND VALUE FOR A SPIN-1 PARTICLE IN A MAGNETIC FIELD

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 10.6.

So far, when calculating phases in the adiabatic theorem we've assumed that there is only one parameter  $R$  in the hamiltonian that is time-dependent. In that case, we can write the geometric phase as

$$\gamma_n(t) = i \int_0^t \left\langle \psi_n(t') \left| \frac{\partial}{\partial t'} \psi_n(t') \right. \right\rangle dt' \quad (1)$$

$$= i \int_{R_i}^{R_f} \left\langle \psi_n \left| \frac{\partial}{\partial R} \psi_n \right. \right\rangle dR \quad (2)$$

and if we take the system through a complete loop where we start at  $R = R_i$  then take  $R$  out to  $R_f$  then back to  $R_i$  again, the net phase is always zero because the two limits on the integral are the same once we've travelled the complete loop.

However, if the hamiltonian has two or more time-dependent parameters, then the chain rule for derivatives says that

$$\frac{\partial \psi_n}{\partial t} = \frac{\partial \psi_n}{\partial R_1} \frac{dR_1}{dt} + \frac{\partial \psi_n}{\partial R_2} \frac{dR_2}{dt} + \dots + \frac{\partial \psi_n}{\partial R_N} \frac{dR_N}{dt} \quad (3)$$

If we treat the complete set of parameters  $R_j$  as the components of an  $N$ -dimensional vector, we can define a gradient in the  $R_j$  coordinate system as  $\nabla_R$  and rewrite this derivative as

$$\frac{\partial \psi_n}{\partial t} = (\nabla_R \psi_n) \cdot \frac{d\mathbf{R}}{dt} \quad (4)$$

giving the phase as

$$\gamma_n(t) = i \int_{\mathbf{R}_i}^{\mathbf{R}_f} \langle \psi_n | \nabla_R \psi_n \rangle \cdot d\mathbf{R} \quad (5)$$

If we now take the system around a closed loop in  $R$ -space in time  $T$ , we can write the phase change over that loop as a line integral around the path:

$$\gamma_n(t) = i \oint \langle \psi_n | \nabla_R \psi_n \rangle \cdot d\mathbf{R} \quad (6)$$

As this is the line integral of a vector around a closed path, if  $\mathbf{R}$  consists of three parameters, we can convert it to a surface integral over the area enclosed by the path by using Stokes's theorem:

$$\gamma_n(t) = i \int (\nabla \times \langle \psi_n | \nabla_R \psi_n \rangle) \cdot d\mathbf{a} \quad (7)$$

This phase is known as **Berry's phase** and is not, in general, zero. Griffiths works out the classic example of calculating Berry's phase for an electron in a precessing magnetic field and then, more generally, for an electron in a magnetic field of constant magnitude but varying in direction by sweeping out some closed path (of any shape). The results apply to any spin-1/2 particle, so here we'll work out Berry's phase for a particle of spin 1.

Ultimately, what we want is to work out  $\gamma$  for some initial spin state of the particle. If we're using spherical coordinates, then  $\nabla_R$  is the usual gradient in spherical coordinates, so to complete the calculation, we need to know  $\psi_n$ . Suppose we start with the particle in the  $+1$  spin state (for spin 1, the  $z$  component can have values  $\pm\hbar$  and 0, so a spin of  $+1$  corresponds to  $+\hbar$ ). The spin matrices are

$$S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (8)$$

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (9)$$

$$S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad (10)$$

so the component of  $\mathbf{S}$  along a direction (which is taken to be the magnetic field's instantaneous direction) given by

$$\hat{\mathbf{r}} = \sin\theta \cos\phi \hat{\mathbf{x}} + \sin\theta \sin\phi \hat{\mathbf{y}} + \cos\theta \hat{\mathbf{z}} \quad (11)$$

is

$$\mathbf{S} \cdot \hat{\mathbf{r}} = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cos \theta + \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \sin \theta \cos \phi + \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \sin \theta \sin \phi \quad (12)$$

$$= \frac{\hbar}{\sqrt{2}} \begin{pmatrix} \sqrt{2} \cos \theta & \sin \theta e^{-i\phi} & 0 \\ \sin \theta e^{i\phi} & 0 & \sin \theta e^{-i\phi} \\ 0 & \sin \theta e^{i\phi} & -\sqrt{2} \cos \theta \end{pmatrix} \quad (13)$$

The eigenvalues of this matrix are  $\pm \hbar, 0$  as required and the normalized eigenvector corresponding to  $+\hbar$  is

$$\chi_{+1} = \frac{1}{2} \begin{bmatrix} e^{-2i\phi} (1 + \cos \theta) \\ \sqrt{2} e^{-i\phi} \sin \theta \\ 1 - \cos \theta \end{bmatrix} \quad (14)$$

The gradient in spherical coordinates of  $\chi_{+1}$  has components only in the  $\theta$  and  $\phi$  directions and we have

$$\nabla \chi_{+1} = \frac{1}{r} \frac{\partial \chi_{+1}}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial \chi_{+1}}{\partial \phi} \hat{\phi} \quad (15)$$

$$= \frac{1}{2r} \begin{bmatrix} -e^{-2i\phi} \sin \theta \\ \sqrt{2} e^{-i\phi} \cos \theta \\ \sin \theta \end{bmatrix} \hat{\theta} - \frac{i}{2r} \begin{bmatrix} \frac{e^{-2i\phi} (1 + \cos \theta)}{\sin \theta} \\ \sqrt{2} e^{-i\phi} \\ 0 \end{bmatrix} \hat{\phi} \quad (16)$$

We can now work out

$$\langle \chi_{+1} | \nabla \chi_{+1} \rangle = 0 \hat{\theta} - i \frac{1 + \cos \theta}{r \sin \theta} \hat{\phi} \quad (17)$$

$$= -i \frac{1 + \cos \theta}{r \sin \theta} \hat{\phi} \quad (18)$$

The curl of this has a component only in the  $r$  direction:

$$\nabla \times \langle \chi_{+1} | \nabla \chi_{+1} \rangle = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left[ -i \frac{1 + \cos \theta}{r \sin \theta} \sin \theta \right] \hat{\mathbf{r}} \quad (19)$$

$$= \frac{i}{r^2} \hat{\mathbf{r}} \quad (20)$$

To get the phase we need to integrate this over the surface enclosed by a complete loop traversed by the point of the  $\mathbf{B}$  field, that is

$$\gamma = i \int \frac{i}{r^2} \hat{\mathbf{r}} \cdot d\mathbf{a} \quad (21)$$

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Since the magnetic field's magnitude is constant, the traversed path is on the surface of a sphere with radius  $r$  and  $d\mathbf{a}$  subtends an element of solid angle on this sphere so that

$$d\mathbf{a} = r^2 \hat{\mathbf{r}} d\Omega \quad (22)$$

The integral thus comes out to

$$\gamma = - \int d\Omega = -\Omega \quad (23)$$