

FORCED HARMONIC OSCILLATOR: EXACT SOLUTION AND ADIABATIC APPROXIMATION

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 10.9.

The one-dimensional driven harmonic oscillator can be solved exactly both classically and in quantum mechanics. If the oscillator's natural frequency is ω , then with a driving force $m\omega^2 f(t)$ where the function f has dimensions of length and can be any function of time (with the condition that $f(t) = 0$ for $t \leq 0$), then the total force is

$$F(t) = m\ddot{x}(t) = m\omega^2 (f(t) - x(t)) \quad (1)$$

The classical solution for $x_c(t)$ (subscript 'c' for 'classical') is given in Griffiths's question as

$$x_c = \omega \int_0^t f(t') \sin[\omega(t-t')] dt' \quad (2)$$

I'm not sure how to solve the original ODE to get this solution, but to show that it *is* a solution, we can work backwards. To do this requires finding the derivative of the integral, which is complicated by the presence of the limit of integration t inside the sine function in the integrand. We can see what the derivative is in the more general case by using the definition of a derivative:

$$I(t) \equiv \int_0^t g(t-t') dt' \quad (3)$$

$$\dot{I} = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left[\int_0^{t+\Delta t} g(t+\Delta t-t') dt' - \int_0^t g(t-t') dt' \right] \quad (4)$$

$$= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left[\int_0^t (g(t+\Delta t-t') - g(t-t')) dt' + \int_t^{t+\Delta t} g(t+\Delta t-t') dt' \right] \quad (5)$$

$$= \int_0^t \frac{dg(t-t')}{dt} dt' + \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_t^{t+\Delta t} g(t+\Delta t-t') dt' \quad (6)$$

In the second term, as $\Delta t \rightarrow 0$, the integral becomes

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_t^{t+\Delta t} g(t+\Delta t-t') dt' &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} [g(t+\Delta t-t)(t+\Delta t-t)] \\ &= g(0) \end{aligned} \quad (8)$$

Therefore

$$\frac{d}{dt} \int_0^t g(t-t') dt' = \int_0^t \frac{dg(t-t')}{dt} dt' + g(0) \quad (9)$$

Applying this to 2 we get

$$\dot{x}_c = \omega^2 \int_0^t f(t') \cos[\omega(t-t')] dt' \quad (10)$$

$$\ddot{x}_c = -\omega^3 \int_0^t f(t') \sin[\omega(t-t')] dt' + \omega^2 f(t) \quad (11)$$

$$= \omega^2 (f(t) - x_c(t)) \quad (12)$$

where in the last line, the function g in 9 is

$$g(t-t') = f(t') \cos[\omega(t-t')] \quad (13)$$

so $g(0)$ occurs when $t = t'$, or

$$g(0) = f(t) \cos(0) = f(t) \quad (14)$$

Thus 2 is actually a solution of 1. The initial conditions are

$$x_c(0) = 0 \quad (15)$$

$$\dot{x}_c(0) = 0 \quad (16)$$

Griffiths now asks us to show that the exact quantum mechanical solution to the Schrödinger equation is

$$\Psi(x, t) = \psi_n(x - x_c) e^{\frac{i}{\hbar} \left[-(n + \frac{1}{2}) \hbar \omega t + m \dot{x}_c (x - \frac{x_c}{2}) + \frac{m \omega^2}{2} \int_0^t f(t') x_c(t') dt' \right]} \quad (17)$$

where ψ_n is the eigenfunction for the unforced oscillator, with eigenvalue $(n + \frac{1}{2}) \hbar \omega$.

The Hamiltonian for the forced oscillator is

$$H(t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 - m \omega^2 x f(t) \quad (18)$$

We can show this by applying H and $i\hbar \frac{\partial}{\partial t}$ to Ψ and showing that they give the same result. We start with the time derivative, remembering that x_c

is a function of time. We'll denote a derivative with respect to t by a dot, and a derivative with respect to x by a dash. We'll also define

$$\eta \equiv e^{\frac{i}{\hbar} \left[-\left(n + \frac{1}{2}\right) \hbar \omega t + m \dot{x}_c \left(x - \frac{x_c}{2}\right) + \frac{m \omega^2}{2} \int_0^t f(t') x_c(t') dt' \right]} \quad (19)$$

We get

$$i \hbar \frac{\partial \Psi}{\partial t} = -i \hbar \dot{x}_c \psi'_n \eta - \psi_n \eta \left[-\left(n + \frac{1}{2}\right) \hbar \omega + m \dot{x}_c \left(x - \frac{x_c}{2}\right) - \frac{m \dot{x}_c^2}{2} + \frac{m \omega^2}{2} f x_c \right] \quad (20)$$

$$= -i \hbar \dot{x}_c \psi'_n \eta - \psi_n \eta \left[-\left(n + \frac{1}{2}\right) \hbar \omega + m \omega^2 (f - x_c) \left(x - \frac{x_c}{2}\right) - \frac{m \dot{x}_c^2}{2} + \frac{m \omega^2}{2} f x_c \right] \quad (21)$$

$$= -i \hbar \dot{x}_c \psi'_n \eta - \psi_n \eta \left[-\left(n + \frac{1}{2}\right) \hbar \omega + m \omega^2 \left(f x - x x_c + \frac{x_c^2}{2}\right) - \frac{m \dot{x}_c^2}{2} \right] \quad (22)$$

where we used 1 in the second line.

Now we apply H to $\Psi(x, t)$. However, the unforced eigenfunction in 17 is given as a function of $x - x_c$, not x , so the Hamiltonian that gives the standard harmonic oscillator eigenvalues is

$$H_0 = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial (x - x_c)^2} + \frac{1}{2} m \omega^2 (x - x_c)^2 \quad (23)$$

$$= -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m \omega^2 (x^2 - 2x x_c + x_c^2) \quad (24)$$

so our forced Hamiltonian is

$$H = H_0 + \frac{1}{2} m \omega^2 (2x x_c - x_c^2) - m \omega^2 x f \quad (25)$$

Applying this to 17 requires finding the second x derivative:

$$-\frac{\hbar^2}{2m} \frac{\partial \Psi}{\partial x} = -\frac{\hbar^2}{2m} \eta \left(\psi'_n + \psi_n \frac{i}{\hbar} m \dot{x}_c \right) \quad (26)$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} = -\frac{\hbar^2}{2m} \eta \left(\psi''_n + 2\psi'_n \left(\frac{i}{\hbar} m \dot{x}_c \right) - \frac{m^2 \dot{x}_c^2}{\hbar^2} \psi_n \right) \quad (27)$$

$$= \eta \left(-\frac{\hbar^2}{2m} \psi''_n - i \hbar \psi'_n \dot{x}_c + \frac{m \dot{x}_c^2}{2} \psi_n \right) \quad (28)$$

Putting it together, we get

$$\begin{aligned}
 H\Psi &= \eta \left(-\frac{\hbar^2}{2m}\psi_n'' - i\hbar\psi_n'\dot{x}_c + \frac{m\dot{x}_c^2}{2}\psi_n \right) + \\
 &\left[\frac{1}{2}m\omega^2(x^2 - 2xx_c + x_c^2) + \frac{1}{2}m\omega^2(2xx_c - x_c^2) - m\omega^2xf \right] \eta\psi_n \quad (29)
 \end{aligned}$$

$$= \eta \left[H_0 + \frac{m\dot{x}_c^2}{2} + \frac{1}{2}m\omega^2(2xx_c - x_c^2) - m\omega^2xf \right] \psi_n - i\hbar\eta\dot{x}_c\psi_n' \quad (30)$$

$$= -i\hbar\dot{x}_c\psi_n'\eta - \psi_n\eta \left[-\left(n + \frac{1}{2}\right)\hbar\omega + m\omega^2\left(fx - xx_c + \frac{x_c^2}{2}\right) - \frac{m\dot{x}_c^2}{2} \right] \quad (31)$$

which is identical to 22, so 17 is indeed a solution of the Schrödinger equation.

Using a similar argument, we can find the eigenfunctions and eigenvalues of the full Hamiltonian. Griffiths gives the eigenfunctions as

$$\psi_n(x, t) = \psi_n(x - f) \quad (32)$$

where $\psi_n(x - f)$ is the unforced eigenfunction evaluated at position $x - f$. We can define an unforced Hamiltonian as above:

$$H_f = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial(x-f)^2} + \frac{1}{2}m\omega^2(x-f)^2 \quad (33)$$

$$= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2(x^2 - 2xf + f^2) \quad (34)$$

The full Hamiltonian becomes

$$H = H_f + \frac{1}{2}m\omega^2(2xf - f^2) - m\omega^2xf \quad (35)$$

$$= H_f - \frac{1}{2}m\omega^2f^2 \quad (36)$$

Applying this to $\psi_n(x - f)$ we get

$$H\psi_n(x - f) = H_f\psi_n(x - f) - \frac{1}{2}m\omega^2f^2\psi_n(x - f) \quad (37)$$

$$= \left[\left(n + \frac{1}{2}\right)\hbar\omega - \frac{1}{2}m\omega^2f^2 \right] \psi_n(x - f) \quad (38)$$

which shows that $\psi_n(x - f)$ is an eigenfunction and the eigenvalues are

$$E_n = \left(n + \frac{1}{2}\right) \hbar\omega - \frac{1}{2}m\omega^2 f^2 \quad (39)$$

Note that both the eigenfunctions and eigenvalues are time-dependent, through the parameter f .

So far, everything has been exact, but we can now apply the adiabatic theorem to the case where the forcing function f varies slowly with time. First, we can return to the classical result 2, and rewrite it as

$$x_c(t) = \omega \int_0^t f(t') \sin[\omega(t-t')] dt' \quad (40)$$

$$= \omega \int_0^t f(t') \frac{1}{\omega} \frac{d}{dt'} \cos[\omega(t-t')] dt' \quad (41)$$

$$= f(t') \cos[\omega(t-t')] \Big|_{t'=0}^{t'=t} - \int_0^t \cos[\omega(t-t')] \frac{d}{dt'} f(t') dt' \quad (42)$$

$$= f(t) - \int_0^t \cos[\omega(t-t')] \frac{d}{dt'} f(t') dt' \quad (43)$$

where we used $f(0) = 0$ in the last line.

Now if f varies slowly compared to the natural frequency ω we can take its derivative outside the integral to get an approximation:

$$x_c(t) \approx f(t) - \dot{f}(t) \int_0^t \cos[\omega(t-t')] dt' \quad (44)$$

$$= f(t) + \frac{\dot{f}(t)}{\omega} \sin[\omega(t-t')] \quad (45)$$

If

$$|\dot{f}(t)| \ll \omega |f(t)| \quad (46)$$

then we neglect the second term to get the classical adiabatic approximation

$$x_c(t) \approx f(t) \quad (47)$$

We can use this approximation to get an adiabatic approximation for the quantum wave function 17:

$$\Psi(x, t) \approx \psi_n(x - f) e^{\frac{i}{\hbar} \left[-\left(n + \frac{1}{2}\right) \hbar\omega t + m f \left(x - \frac{f}{2}\right) + \frac{m\omega^2}{2} \int_0^t f^2(t') dt' \right]} \quad (48)$$

$$= \psi_n(x - f) e^{i\theta_n(t)} e^{i\gamma_n(t)} \quad (49)$$

where

$$\theta_n(t) = \frac{m\omega^2}{2\hbar} \int_0^t f^2(t') dt' - \left(n + \frac{1}{2}\right) \omega t \quad (50)$$

$$\gamma_n(t) = \frac{m\dot{f}}{\hbar} \left(x - \frac{f}{2}\right) \quad (51)$$

Here the phase factors are θ (the dynamic phase) and γ (the geometric phase). From 39 we see that

$$\theta_n(t) = -\frac{1}{\hbar} \int_0^t E_n(t') dt' \quad (52)$$

which agrees with its earlier definition.

If the eigenfunctions are real, then the geometric phase should be zero. This isn't strictly true here, but we're assuming that \dot{f} is small, so γ_n should be close to zero.

PINGBACKS

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