FORCED HARMONIC OSCILLATOR: EXACT SOLUTION AND ADIABATIC APPROXIMATION

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 10.9.

The one-dimensional driven harmonic oscillator can be solved exactly both classically and in quantum mechanics. If the oscillator's natural frequency is ω, then with a driving force $m\omega^2 f(t)$ where the function f has dimensions of length and can be any function of time (with the condition that $f(t)=0$ for $t\le 0$), then the total force is

$$F(t) = m\ddot{x}(t) = m\omega^2 (f(t) - x(t)) \tag{0.1}$$

The classical solution for $x_c(t)$ (subscript 'c' for 'classical') is given in Griffiths's question as

$$x_c = \omega \int_0^t f(t') \sin\left[\omega(t-t')\right] dt' \tag{0.2}$$

I'm not sure how to solve the original ODE to get this solution, but to show that it *is* a solution, we can work backwards. To do this requires finding the derivative of the integral, which is complicated by the presence of the limit of integration t inside the sine function in the integrand. We can see what the derivative is in the more general case by using the definition of a derivative:

(0.3)

$$I(t) \equiv \int_0^t g(t-t')\,dt'$$

(0.4)

$$\dot{I} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_0^{t+\Delta t} g(t+\Delta t-t')\,dt' - \int_0^t g(t-t')\,dt'\right]$$

(0.5)

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_0^{t}\left(g(t+\Delta t-t') - g(t-t')\right)dt' + \int_t^{t+\Delta t} g(t+\Delta t-t')\,dt'\right]$$

(0.6)

$$= \int_0^t \frac{dg(t-t')}{dt}dt' + \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_t^{t+\Delta t} g(t+\Delta t-t')\,dt'$$

In the second term, as $\Delta t \to 0$, the integral becomes

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_t^{t+\Delta t} g(t+\Delta t-t')\,dt' = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[g(t+\Delta t-t)(t+\Delta t-t)\right] \tag{0.7}$$

$$= g(0) \tag{0.8}$$

Therefore

$$\frac{d}{dt}\int_0^t g(t-t')\,dt' = \int_0^t \frac{dg(t-t')}{dt}dt' + g(0) \tag{0.9}$$

Applying this to 0.2 we get

$$\dot{x}_c = \omega^2\int_0^t f(t')\cos\left[\omega(t-t')\right]dt' \tag{0.10}$$

$$\ddot{x}_c = -\omega^3\int_0^t f(t')\sin\left[\omega(t-t')\right]dt' + \omega^2 f(t) \tag{0.11}$$

$$= \omega^2\left(f(t) - x_c(t)\right) \tag{0.12}$$

where in the last line, the function g in 0.9 is

$$g(t-t') = f(t')\cos\left[\omega(t-t')\right] \tag{0.13}$$

so $g(0)$ occurs when $t = t'$, or

$$g(0) = f(t)\cos(0) = f(t) \tag{0.14}$$

Thus 0.2 is actually a solution of 0.1. The initial conditions are

$$x_c(0) = 0 \tag{0.15}$$
$$\dot{x}_c(0) = 0 \tag{0.16}$$

Griffiths now asks us to show that the exact quantum mechanical solution to the Schrödinger equation is

$$\Psi(x,t) = \psi_n(x-x_c)\, e^{\frac{i}{\hbar}\left[-\left(n+\frac{1}{2}\right)\hbar\omega t + m\dot{x}_c\left(x-\frac{x_c}{2}\right)+\frac{m\omega^2}{2}\int_0^t f(t')x_c(t')dt'\right]} \tag{0.17}$$

where ψ_n is the eigenfunction for the unforced oscillator, with eigenvalue $\left(n+\frac{1}{2}\right)\hbar\omega$.

The Hamiltonian for the forced oscillator is

$$H(t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2 - m\omega^2 x f(t) \tag{0.18}$$

We can show this by applying H and $i\hbar\frac{\partial}{\partial t}$ to Ψ and showing that they give the same result. We start with the time derivative, remembering that x_c is a function of time. We'll denote a derivative with respect to t by a dot, and a derivative with respect to x by a dash. We'll also define

$$\eta \equiv e^{\frac{i}{\hbar}\left[-\left(n+\frac{1}{2}\right)\hbar\omega t + m\dot{x}_c\left(x-\frac{x_c}{2}\right)+\frac{m\omega^2}{2}\int_0^t f(t')x_c(t')dt'\right]} \tag{0.19}$$

We get

$$i\hbar\frac{\partial\Psi}{\partial t} = -i\hbar\dot{x}_c\psi_n'\eta - \psi_n\eta\left[-\left(n+\frac{1}{2}\right)\hbar\omega + m\ddot{x}_c\left(x-\frac{x_c}{2}\right) - \frac{m\dot{x}_c^2}{2} + \frac{m\omega^2}{2}fx_c\right] \tag{0.20}$$

$$= -i\hbar\dot{x}_c\psi_n'\eta - \psi_n\eta\left[-\left(n+\frac{1}{2}\right)\hbar\omega + m\omega^2(f-x_c)\left(x-\frac{x_c}{2}\right) - \frac{m\dot{x}_c^2}{2} + \frac{m\omega^2}{2}fx_c\right] \tag{0.21}$$

$$= -i\hbar\dot{x}_c\psi_n'\eta - \psi_n\eta\left[-\left(n+\frac{1}{2}\right)\hbar\omega + m\omega^2\left(fx - xx_c + \frac{x_c^2}{2}\right) - \frac{m\dot{x}_c^2}{2}\right] \tag{0.22}$$

where we used 0.1 in the second line.

Now we apply H to $\Psi(x,t)$. However, the unforced eigenfunction in 0.17 is given as a function of $x-x_c$, not x, so the Hamiltonian that gives the standard harmonic oscillator eigenvalues is

$$H_0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\left(x-x_c\right)^2} + \frac{1}{2}m\omega^2\left(x-x_c\right)^2 \tag{0.23}$$

$$= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2\left(x^2-2xx_c+x_c^2\right) \tag{0.24}$$

so our forced Hamiltonian is

$$H = H_0 + \frac{1}{2}m\omega^2\left(2xx_c - x_c^2\right) - m\omega^2 xf \tag{0.25}$$

Applying this to 0.17 requires finding the second x derivative:

$$-\frac{\hbar^2}{2m}\frac{\partial\Psi}{\partial x} = -\frac{\hbar^2}{2m}\eta\left(\psi_n' + \psi_n\frac{i}{\hbar}m\dot{x}_c\right) \tag{0.26}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = -\frac{\hbar^2}{2m}\eta\left(\psi_n'' + 2\psi_n'\left(\frac{i}{\hbar}m\dot{x}_c\right) - \frac{m^2\dot{x}_c^2}{\hbar^2}\psi_n\right) \tag{0.27}$$

$$= \eta\left(-\frac{\hbar^2}{2m}\psi_n'' - i\hbar\psi_n'\dot{x}_c + \frac{m\dot{x}_c^2}{2}\psi_n\right) \tag{0.28}$$

Putting it together, we get

$$H\Psi = \eta\left(-\frac{\hbar^2}{2m}\psi_n'' - i\hbar\psi_n'\dot{x}_c + \frac{m\dot{x}_c^2}{2}\psi_n\right) + \tag{0.29}$$

$$\left[\frac{1}{2}m\omega^2\left(x^2-2xx_c+x_c^2\right) + \frac{1}{2}m\omega^2\left(2xx_c-x_c^2\right) - m\omega^2 xf\right]\eta\psi_n \tag{0.30}$$

$$= \eta\left[H_0 + \frac{m\dot{x}_c^2}{2} + \frac{1}{2}m\omega^2\left(2xx_c-x_c^2\right) - m\omega^2 xf\right]\psi_n - i\hbar\eta\dot{x}_c\psi_n' \tag{0.31}$$

$$= -i\hbar\dot{x}_c\psi_n'\eta - \psi_n\eta\left[-\left(n+\frac{1}{2}\right)\hbar\omega + m\omega^2\left(fx - xx_c + \frac{x_c^2}{2}\right) - \frac{m\dot{x}_c^2}{2}\right]$$

which is identical to 0.22, so 0.17 is indeed a solution of the Schrödinger equation.

Using a similar argument, we can find the eigenfunctions and eigenvalues of the full Hamiltonian. Griffiths gives the eigenfunctions as

$$\psi_n\left(x,t\right) = \psi_n\left(x-f\right) \tag{0.32}$$

where $\psi_n(x-f)$ is the unforced eigenfunction evaluated at position $x-f$. We can define an unforced Hamiltonian as above:

$$H_f = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial(x-f)^2} + \frac{1}{2}m\omega^2(x-f)^2 \tag{0.33}$$

$$= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2\left(x^2-2xf+f^2\right) \tag{0.34}$$

The full Hamiltonian becomes

$$H = H_f + \frac{1}{2}m\omega^2\left(2xf-f^2\right) - m\omega^2 xf \tag{0.35}$$

$$= H_f - \frac{1}{2}m\omega^2 f^2 \tag{0.36}$$

Applying this to $\psi_n(x-f)$ we get

$$H\psi_n(x-f) = H_f\psi_n(x-f) - \frac{1}{2}m\omega^2 f^2\psi_n(x-f) \tag{0.37}$$

$$= \left[\left(n+\frac{1}{2}\right)\hbar\omega - \frac{1}{2}m\omega^2 f^2\right]\psi_n(x-f) \tag{0.38}$$

which shows that $\psi_n(x-f)$ is an eigenfunction and the eigenvalues are

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega - \frac{1}{2}m\omega^2 f^2 \tag{0.39}$$

Note that both the eigenfunctions and eigenvalues are time-dependent, through the parameter f.

So far, everything has been exact, but we can now apply the adiabatic theorem to the case where the forcing function f varies slowly with time. First, we can return to the classical result 0.2, and rewrite it as

$$x_c(t) = \omega\int_0^t f(t')\sin\left[\omega(t-t')\right]dt' \tag{0.40}$$

$$= \omega\int_0^t f(t')\frac{1}{\omega}\frac{d}{dt'}\cos\left[\omega(t-t')\right]dt' \tag{0.41}$$

$$= f(t')\cos\left[\omega(t-t')\right]\Big|_{t'=0}^{t'=t} - \int_0^t \cos\left[\omega(t-t')\right]\frac{d}{dt'}f(t')\,dt' \tag{0.42}$$

$$= f(t) - \int_0^t \cos\left[\omega(t-t')\right]\frac{d}{dt'}f(t')\,dt' \tag{0.43}$$

where we used $f(0)=0$ in the last line.

Now if f varies slowly compared to the natural frequency ω we can take its derivative outside the integral to get an approximation:

$$x_c(t) \approx f(t)-\dot{f}(t)\int_0^t \cos\left[\omega\left(t-t'\right)\right]dt' \tag{0.44}$$

$$= f(t)+\frac{\dot{f}(t)}{\omega}\sin\left[\omega\left(t-t'\right)\right] \tag{0.45}$$

If

$$\left|\dot{f}(t)\right| \ll \omega\left|f(t)\right| \tag{0.46}$$

then we neglect the second term to get the classical adiabatic approximation

$$x_c(t)\approx f(t) \tag{0.47}$$

We can use this approximation to get an adiabatic approximation for the quantum wave function 0.17:

$$\Psi(x,t)\approx \psi_n(x-f)\,e^{\frac{i}{\hbar}\left[-\left(n+\frac{1}{2}\right)\hbar\omega t+m\dot{f}\left(x-\frac{f}{2}\right)+\frac{m\omega^2}{2}\int_0^t f^2(t')dt'\right]} \tag{0.48}$$

$$=\psi_n(x-f)\,e^{i\theta_n(t)}e^{i\gamma_n(t)} \tag{0.49}$$

where

$$\theta_n(t) = \frac{m\omega^2}{2\hbar}\int_0^t f^2\left(t'\right)dt'-\left(n+\frac{1}{2}\right)\omega t \tag{0.50}$$

$$\gamma_n(t) = \frac{m\dot{f}}{\hbar}\left(x-\frac{f}{2}\right) \tag{0.51}$$

Here the phase factors are θ (the dynamic phase) and γ (the geometric phase). From 0.39 we see that

$$\theta_n(t)=-\frac{1}{\hbar}\int_0^t E_n\left(t'\right)dt' \tag{0.52}$$

which agrees with its earlier definition.

If the eigenfunctions are real, then the geometric phase should be zero. This isn't strictly true here, but we're assuming that $\dot{f}$ is small, so γ_n should be close to zero.

PINGBACKS

Pingback: Adiabatic approximation: higher order corrections