FORCED HARMONIC OSCILLATOR: EXACT SOLUTION AND ADIABATIC APPROXIMATION

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 10.9.

The one-dimensional driven harmonic oscillator can be solved exactly both classically and in quantum mechanics. If the oscillator's natural frequency is ω, then with a driving force $m\omega^2 f(t)$ where the function f has dimensions of length and can be any function of time (with the condition that $f(t)=0$ for $t\le 0$), then the total force is

$$F(t)=m\ddot{x}(t)=m\omega^2\left(f(t)-x(t)\right) \tag{1}$$

The classical solution for $x_c(t)$ (subscript 'c' for 'classical') is given in Griffiths's question as

$$x_c=\omega\int_0^t f\left(t'\right)\sin\left[\omega\left(t-t'\right)\right]dt' \tag{2}$$

I'm not sure how to solve the original ODE to get this solution, but to show that it *is* a solution, we can work backwards. To do this requires finding the derivative of the integral, which is complicated by the presence of the limit of integration t inside the sine function in the integrand. We can see what the derivative is in the more general case by using the definition of a derivative:

$$I(t)\equiv\int_0^t g\left(t-t'\right)dt' \tag{3}$$

$$\dot{I}=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_0^{t+\Delta t}g\left(t+\Delta t-t'\right)dt'-\int_0^t g\left(t-t'\right)dt'\right] \tag{4}$$

$$=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_0^t\left(g\left(t+\Delta t-t'\right)-g\left(t-t'\right)\right)dt'+\int_t^{t+\Delta t}g\left(t+\Delta t-t'\right)dt'\right] \tag{5}$$

$$=\int_0^t\frac{dg\left(t-t'\right)}{dt}dt'+\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_t^{t+\Delta t}g\left(t+\Delta t-t'\right)dt' \tag{6}$$

In the second term, as $\Delta t\to 0$, the integral becomes

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{t}^{t+\Delta t}g\left(t+\Delta t-t'\right)dt' = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[g\left(t+\Delta t-t\right)\left(t+\Delta t-t\right)\right] \tag{7}$$

$$= g\left(0\right) \tag{8}$$

Therefore

$$\frac{d}{dt}\int_{0}^{t}g\left(t-t'\right)dt' = \int_{0}^{t}\frac{dg\left(t-t'\right)}{dt}dt'+g\left(0\right) \tag{9}$$

Applying this to 2 we get

$$\dot{x}_c = \omega^2\int_0^t f\left(t'\right)\cos\left[\omega\left(t-t'\right)\right]dt' \tag{10}$$

$$\ddot{x}_c = -\omega^3\int_0^t f\left(t'\right)\sin\left[\omega\left(t-t'\right)\right]dt'+\omega^2 f\left(t\right) \tag{11}$$

$$= \omega^2\left(f\left(t\right)-x_c\left(t\right)\right) \tag{12}$$

where in the last line, the function g in 9 is

$$g\left(t-t'\right) = f\left(t'\right)\cos\left[\omega\left(t-t'\right)\right] \tag{13}$$

so $g\left(0\right)$ occurs when $t=t'$, or

$$g\left(0\right) = f\left(t\right)\cos\left(0\right) = f\left(t\right) \tag{14}$$

Thus 2 is actually a solution of 1. The initial conditions are

$$x_c\left(0\right) = 0 \tag{15}$$

$$\dot{x}_c\left(0\right) = 0 \tag{16}$$

Griffiths now asks us to show that the exact quantum mechanical solution to the Schrödinger equation is

$$\Psi\left(x,t\right) = \psi_n\left(x-x_c\right)e^{\frac{i}{\hbar}\left[-\left(n+\frac{1}{2}\right)\hbar\omega t+m\dot{x}_c\left(x-\frac{x_c}{2}\right)+\frac{m\omega^2}{2}\int_0^t f\left(t'\right)x_c\left(t'\right)dt'\right]} \tag{17}$$

where ψ_n is the eigenfunction for the unforced oscillator, with eigenvalue $\left(n+\frac{1}{2}\right)\hbar\omega$.

The Hamiltonian for the forced oscillator is

$$H\left(t\right) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{1}{2}m\omega^2x^2-m\omega^2xf\left(t\right) \tag{18}$$

We can show this by applying H and $i\hbar\frac{\partial}{\partial t}$ to Ψ and showing that they give the same result. We start with the time derivative, remembering that x_c

is a function of time. We'll denote a derivative with respect to t by a dot, and a derivative with respect to x by a dash. We'll also define

$$\eta \equiv e^{\frac{i}{\hbar}\left[-\left(n+\frac{1}{2}\right)\hbar\omega t+m\dot{x}_c\left(x-\frac{x_c}{2}\right)+\frac{m\omega^2}{2}\int_0^t f(t')x_c(t')dt'\right]} \tag{19}$$

We get

$$i\hbar\frac{\partial\Psi}{\partial t} = -i\hbar\dot{x}_c\psi_n'\eta-\psi_n\eta\left[-\left(n+\frac{1}{2}\right)\hbar\omega+m\ddot{x}_c\left(x-\frac{x_c}{2}\right)-\frac{m\dot{x}_c^2}{2}+\frac{m\omega^2}{2}fx_c\right] \tag{20}$$

$$= -i\hbar\dot{x}_c\psi_n'\eta-\psi_n\eta\left[-\left(n+\frac{1}{2}\right)\hbar\omega+m\omega^2\left(f-x_c\right)\left(x-\frac{x_c}{2}\right)-\frac{m\dot{x}_c^2}{2}+\frac{m\omega^2}{2}fx_c\right] \tag{21}$$

$$= -i\hbar\dot{x}_c\psi_n'\eta-\psi_n\eta\left[-\left(n+\frac{1}{2}\right)\hbar\omega+m\omega^2\left(fx-xx_c+\frac{x_c^2}{2}\right)-\frac{m\dot{x}_c^2}{2}\right] \tag{22}$$

where we used 1 in the second line.

Now we apply H to $\Psi(x,t)$. However, the unforced eigenfunction in 17 is given as a function of $x-x_c$, not x, so the Hamiltonian that gives the standard harmonic oscillator eigenvalues is

$$H_0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\left(x-x_c\right)^2}+\frac{1}{2}m\omega^2\left(x-x_c\right)^2 \tag{23}$$

$$= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{1}{2}m\omega^2\left(x^2-2xx_c+x_c^2\right) \tag{24}$$

so our forced Hamiltonian is

$$H = H_0+\frac{1}{2}m\omega^2\left(2xx_c-x_c^2\right)-m\omega^2xf \tag{25}$$

Applying this to 17 requires finding the second x derivative:

$$-\frac{\hbar^2}{2m}\frac{\partial\Psi}{\partial x} = -\frac{\hbar^2}{2m}\eta\left(\psi_n'+\psi_n\frac{i}{\hbar}m\dot{x}_c\right) \tag{26}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = -\frac{\hbar^2}{2m}\eta\left(\psi_n''+2\psi_n'\left(\frac{i}{\hbar}m\dot{x}_c\right)-\frac{m^2\dot{x}_c^2}{\hbar^2}\psi_n\right) \tag{27}$$

$$= \eta\left(-\frac{\hbar^2}{2m}\psi_n''-i\hbar\psi_n'\dot{x}_c+\frac{m\dot{x}_c^2}{2}\psi_n\right) \tag{28}$$

Putting it together, we get

$$
\begin{aligned}
H\Psi &= \eta\left(-\frac{\hbar^2}{2m}\psi_n'' - i\hbar\psi_n'\dot{x}_c + \frac{m\dot{x}_c^2}{2}\psi_n\right) + \\
&\quad \left[\frac{1}{2}m\omega^2\left(x^2 - 2xx_c + x_c^2\right) + \frac{1}{2}m\omega^2\left(2xx_c - x_c^2\right) - m\omega^2 xf\right]\eta\psi_n && (29)\\
&= \eta\left[H_0 + \frac{m\dot{x}_c^2}{2} + \frac{1}{2}m\omega^2\left(2xx_c - x_c^2\right) - m\omega^2 xf\right]\psi_n - i\hbar\eta\dot{x}_c\psi_n' && (30)\\
&= -i\hbar\dot{x}_c\psi_n'\eta - \psi_n\eta\left[-\left(n+\frac{1}{2}\right)\hbar\omega + m\omega^2\left(fx - xx_c + \frac{x_c^2}{2}\right) - \frac{m\dot{x}_c^2}{2}\right] && (31)
\end{aligned}
$$

which is identical to 22, so 17 is indeed a solution of the Schrödinger equation.

Using a similar argument, we can find the eigenfunctions and eigenvalues of the full Hamiltonian. Griffiths gives the eigenfunctions as

$$\psi_n(x,t) = \psi_n(x-f) \tag{32}$$

where $\psi_n(x-f)$ is the unforced eigenfunction evaluated at position $x-f$. We can define an unforced Hamiltonian as above:

$$
\begin{aligned}
H_f &= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial(x-f)^2} + \frac{1}{2}m\omega^2(x-f)^2 && (33)\\
&= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2\left(x^2 - 2xf + f^2\right) && (34)
\end{aligned}
$$

The full Hamiltonian becomes

$$
\begin{aligned}
H &= H_f + \frac{1}{2}m\omega^2\left(2xf - f^2\right) - m\omega^2 xf && (35)\\
&= H_f - \frac{1}{2}m\omega^2 f^2 && (36)
\end{aligned}
$$

Applying this to $\psi_n(x-f)$ we get

$$
\begin{aligned}
H\psi_n(x-f) &= H_f\psi_n(x-f) - \frac{1}{2}m\omega^2 f^2\psi_n(x-f) && (37)\\
&= \left[\left(n+\frac{1}{2}\right)\hbar\omega - \frac{1}{2}m\omega^2 f^2\right]\psi_n(x-f) && (38)
\end{aligned}
$$

which shows that $\psi_n(x-f)$ is an eigenfunction and the eigenvalues are

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega - \frac{1}{2}m\omega^2 f^2 \tag{39}$$

Note that both the eigenfunctions and eigenvalues are time-dependent, through the parameter f.

So far, everything has been exact, but we can now apply the adiabatic theorem to the case where the forcing function f varies slowly with time. First, we can return to the classical result 2, and rewrite it as

$$x_c(t) = \omega\int_0^t f(t')\sin\left[\omega(t-t')\right]dt' \tag{40}$$

$$= \omega\int_0^t f(t')\frac{1}{\omega}\frac{d}{dt'}\cos\left[\omega(t-t')\right]dt' \tag{41}$$

$$= f(t')\cos\left[\omega(t-t')\right]\Big|_{t'=0}^{t'=t} - \int_0^t \cos\left[\omega(t-t')\right]\frac{d}{dt'}f(t')\,dt' \tag{42}$$

$$= f(t) - \int_0^t \cos\left[\omega(t-t')\right]\frac{d}{dt'}f(t')\,dt' \tag{43}$$

where we used $f(0) = 0$ in the last line.

Now if f varies slowly compared to the natural frequency ω we can take its derivative outside the integral to get an approximation:

$$x_c(t) \approx f(t) - \dot{f}(t)\int_0^t \cos\left[\omega(t-t')\right]dt' \tag{44}$$

$$= f(t) + \frac{\dot{f}(t)}{\omega}\sin\left[\omega(t-t')\right] \tag{45}$$

If

$$\left|\dot{f}(t)\right| \ll \omega\left|f(t)\right| \tag{46}$$

then we neglect the second term to get the classical adiabatic approximation

$$x_c(t) \approx f(t) \tag{47}$$

We can use this approximation to get an adiabatic approximation for the quantum wave function 17:

$$\Psi(x,t) \approx \psi_n(x-f)\,e^{\frac{i}{\hbar}\left[-\left(n+\frac{1}{2}\right)\hbar\omega t + m\dot{f}\left(x-\frac{f}{2}\right) + \frac{m\omega^2}{2}\int_0^t f^2(t')dt'\right]} \tag{48}$$

$$= \psi_n(x-f)\,e^{i\theta_n(t)}e^{i\gamma_n(t)} \tag{49}$$

where

$$\theta_n(t) = \frac{m\omega^2}{2\hbar}\int_0^t f^2(t')\,dt' - \left(n+\frac{1}{2}\right)\omega t \tag{50}$$

$$\gamma_n(t) = \frac{m\dot{f}}{\hbar}\left(x-\frac{f}{2}\right) \tag{51}$$

Here the phase factors are θ (the dynamic phase) and γ (the geometric phase). From 39 we see that

$$\theta_n(t) = -\frac{1}{\hbar}\int_0^t E_n(t')\,dt' \tag{52}$$

which agrees with its earlier definition.

If the eigenfunctions are real, then the geometric phase should be zero. This isn't strictly true here, but we're assuming that $\dot{f}$ is small, so γ_n should be close to zero.

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