ADIABATIC APPROXIMATION: HIGHER ORDER CORRECTIONS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 10.10.

In deriving the adiabatic theorem, Griffiths (in his section 10.1) shows that the solution to the time-dependent Schrödinger equation can be written as

$$\Psi(x,t)=\sum_n c_n(t)\,\psi_n(x,t)\,e^{i\theta_n(t)} \tag{1}$$

where the ψ_n form an orthonormal set of functions that are eigenfunctions of the Hamiltonian at a particular instant of time, and θ_n is the dynamic phase. The coefficients c_n are the usual weighting factors, and they depend only on time.

Later in the derivation, he arrives at a differential equation for the c_m:

$$\dot{c}_m(t)=-\sum_j c_j\left\langle\psi_m\left|\dot{\psi}_j\right.\right\rangle e^{i\left(\theta_j-\theta_m\right)} \tag{2}$$

In the adiabatic approximation, this equation has the approximate solution

$$\begin{aligned}
c_m(t) &= c_m(0)\,e^{i\gamma_m(t)} &(3)\\
\gamma_m(t) &\equiv i\int_0^t\left\langle\psi_m\left(t'\right)\left|\frac{\partial}{\partial t'}\psi_m\left(t'\right)\right.\right\rangle dt' &(4)
\end{aligned}$$

where γ_m is the geometric phase. In particular, if the system starts in a definite eigenstate ψ_n then $c_m(0)=\delta_{nm}$ so

$$c_m(t)=\delta_{nm}e^{i\gamma_n(t)} \tag{5}$$

with the result that the overall solution becomes

$$\Psi_n(x,t)=\psi_n(x,t)\,e^{i\theta_n(t)}e^{i\gamma_n(t)} \tag{6}$$

that is, the system stays in the n^{th} state over time, although its phase can change.

We can extend the adiabatic approximation recursively by using the first approximation 5 to generate the next approximation. We can do this by inserting 5 into 2 and then solving the resulting differential equation. The sum in 2 is reduced to a single term where $j=n$, the eigenstate in which the system starts at $t=0$.

$$\dot{c}_m(t) = -e^{i\gamma_n(t)}\langle\psi_m|\dot{\psi}_n\rangle e^{i(\theta_n-\theta_m)} \tag{7}$$

$$c_m(t) = c_m(0) - \int_0^t e^{i\gamma_n(t)}\langle\psi_m|\dot{\psi}_n\rangle e^{i(\theta_n-\theta_m)}dt' \tag{8}$$

This correction to the basic adiabatic approximation now has the ability to predict transitions from the initial state ψ_n to other states ψ_m where $m\neq n$. We can apply this to the forced oscillator, where we found that in the adiabatic approximation

$$\psi_n(x,t) = \psi_n(x-f) \tag{9}$$

$$\theta_n(t) = \frac{m\omega^2}{2\hbar}\int_0^t f^2(t')\,dt' - \left(n+\frac{1}{2}\right)\omega t \tag{10}$$

$$\gamma_n(t) = \frac{m\dot{f}}{\hbar}\left(x-\frac{f}{2}\right)\approx 0 \tag{11}$$

Here, $m\omega^2 f(t)$ is the forcing term, and the adiabatic approximation is obtained by assuming that f changes very slowly, or to be precise:

$$\left|\dot{f}(t)\right| \ll \omega\left|f(t)\right| \tag{12}$$

To work out the correction, we need to find $\langle\psi_m|\dot{\psi}_n\rangle$ in 8. We can do this using the raising and lowering operators for the harmonic oscillator. In particular, the momentum operator can be written in terms of them as

$$p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-) \tag{13}$$

Also, recall that the effects of $a_\pm$ are

$$a_+\psi_n = \sqrt{n+1}\,\psi_n \tag{14}$$

$$a_-\psi_n = \sqrt{n}\,\psi_{n-1} \tag{15}$$

How does this help us? We need to find the derivative $\partial\psi_n(x-f)/\partial t'$, so we get, defining $z\equiv x-f$:

$$\begin{aligned}
\frac{\partial\psi_n(x-f)}{\partial t'} &= \frac{\partial\psi_n(z)}{\partial z}\frac{\partial z}{\partial t'} && (16)\\
&= -\frac{\partial\psi_n(z)}{\partial z}\dot{f} && (17)\\
&= -\frac{\partial\psi_n}{\partial x}\dot{f} && (18)
\end{aligned}$$

where the last line follows because $z=x-f$ and f doesn't depend on x.
Now the momentum operator is

$$p=\frac{\hbar}{i}\frac{\partial}{\partial x} \tag{19}$$

so our derivative is

$$\begin{aligned}
\frac{\partial\psi_n(x-f)}{\partial t'} &= -\frac{i}{\hbar}\dot{f}p\psi_n && (20)\\
&= \dot{f}\sqrt{\frac{m\omega}{2\hbar}}\left(a_+-a_-\right)\psi_n && (21)\\
&= \dot{f}\sqrt{\frac{m\omega}{2\hbar}}\left(\sqrt{n+1}\psi_{n+1}-\sqrt{n}\psi_{n-1}\right) && (22)
\end{aligned}$$

Using the orthonormality of the ψ_m we have

$$\begin{aligned}
\langle\psi_{n+1}|\dot{\psi}_n\rangle &= \dot{f}\sqrt{\frac{m\omega}{2\hbar}}\sqrt{n+1} && (23)\\
\langle\psi_{n-1}|\dot{\psi}_n\rangle &= -\dot{f}\sqrt{\frac{m\omega}{2\hbar}}\sqrt{n} && (24)
\end{aligned}$$

with all other matrix elements equal to zero.
Returning to 8 we can work out the phase terms from 10 and 11.

$$\begin{aligned}
\gamma_n &\approx 0 && (25)\\
\theta_n-\theta_{n+1} &= \omega t && (26)\\
\theta_n-\theta_{n-1} &= -\omega t && (27)
\end{aligned}$$

Therefore, since $c_{n+1}(0)=c_{n-1}(0)=0$,

$$c_{n+1}(t) = -\sqrt{\frac{m\omega}{2\hbar}}\sqrt{n+1}\int_0^t \dot{f}e^{i\omega t'}dt' \tag{28}$$

$$c_{n-1}(t) = \sqrt{\frac{m\omega}{2\hbar}}\sqrt{n}\int_0^t \dot{f}e^{-i\omega t'}dt' \tag{29}$$

[These answers aren't the same as those given in Griffiths's question (although the square moduli are the same) but I can't see anything wrong with my derivation. Comments welcome.]

Note that

$$\langle\psi_n|\dot{\psi}_n\rangle = 0 \tag{30}$$

so 8 predicts that $c_n(t) = c_n(0) = 1$, thus the sum of the square moduli of the c_ms is greater than 1. However, these values for the c_ms are correct only to first order in $\dot{f}$. To get the second order corrections, we'd need to insert 28 and 29 back into 2 and integrate again to get new values for the c_ms, which would give $c_n(t) < 1$ for $t > 0$. The process can be continued as long as we like, giving an adiabatic series.