RUTHERFORD SCATTERING

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 11.1.

Thornton, Stephen T. & Marion, Jerry B. (2004), Classical Dynamics of Particles and Systems, 5th edition; Thomson Brooks/Cole - Chapters 8 and 9.

As a prelude to studying quantum scattering, we'll begin by looking at a classical scattering problem. The particular problem we'll examine is *Rutherford scattering*, which was used by Rutherford in his classic experiments to determine the nature of the atom.

We'll consider Coulomb scattering in the ideal case where a particle with charge q_1 and mass m is fired at a much heavier (and thus stationary) target with charge q_2. The starting point is a consideration of energy and angular momentum. First we need to fix a few quantities.

Suppose the incident particle comes in with an energy E which is initially entirely kinetic (that is, the particle is far enough from the target that the Coulomb potential energy is negligible). We assume it has an *impact parameter* b (that is, if the particle didn't interact with the target, it would pass by with a closest approach distance of b). The angular momentum of the particle relative to the target is

$$\mathbf{L} = m\mathbf{r}\times\mathbf{v} \tag{1}$$

If the particle is coming in parallel to the z axis and the target is at the origin, then the cross product isolates the component of $\mathbf{v}$ that is perpendicular to $\mathbf{r}$, which has magnitude $r\dot{\theta}$, so

$$L = mr^2\dot{\theta} \tag{2}$$

The total energy of the particle is the sum of its kinetic energy parallel to and perpendicular to $\mathbf{r}$, together with its potential energy, so

$$\begin{aligned} E &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{q_1q_2}{4\pi\epsilon_0 r} && (3) \\ &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{L^2}{mr^2} + \frac{q_1q_2}{4\pi\epsilon_0 r} && (4) \end{aligned}$$

Solving for $\dot{r}$ we get

$$\dot{r} = \pm\sqrt{\frac{2}{m}\left(E - \frac{q_1 q_2}{4\pi\epsilon_0 r}\right) - \frac{L^2}{mr^2}} \tag{5}$$

We'd like an expression for the angle in terms of r so we need to get rid of the time derivatives. Using the chain rule, we have

$$d\theta = \frac{d\theta}{dt}\frac{dt}{dr}dr = \frac{\dot{\theta}}{\dot{r}}dr \tag{6}$$

so

$$\begin{aligned}
\theta(r) &= \pm\int \frac{L}{mr^2\sqrt{\frac{2}{m}\left(E - \frac{q_1 q_2}{4\pi\epsilon_0 r}\right) - \frac{L^2}{mr^2}}}dr && (7)\\
&= \pm\frac{L}{\sqrt{2m}}\int \frac{dr}{r^2\sqrt{E - \frac{q_1 q_2}{4\pi\epsilon_0 r} - \frac{L^2}{2mr^2}}} && (8)\\
&= \pm\frac{L}{\sqrt{2m}}\int \frac{dr}{r\sqrt{Er^2 - \frac{q_1 q_2}{4\pi\epsilon_0} r - \frac{L^2}{2m}}} && (9)
\end{aligned}$$

(Recall L and E are both constants of the motion.) When the particle is far away, its speed is given by

$$v = \sqrt{\frac{2E}{m}} \tag{10}$$

If there were no interaction, the particle would pass by the target with this speed at a closest distance b, so its angular momentum is

$$L = mbv = b\sqrt{2mE} \tag{11}$$

Inserting this into 9 we get

$$\begin{aligned}
\theta(r) &= b\sqrt{E}\int \frac{dr}{r\sqrt{E}\sqrt{r^2 - \frac{q_1 q_2}{4\pi\epsilon_0 E} r - b^2}} && (12)\\
&= b\int \frac{dr}{r\sqrt{r^2 - \frac{q_1 q_2}{4\pi\epsilon_0 E} r - b^2}} && (13)
\end{aligned}$$

We now need to put limits on the integral. Ultimately, we want the angle Θ between the particle's incident path and its asymptotic outgoing path. (Notation alert: I use θ for the angle measured relative to the target and Θ for the angle measured relative to the particle's incident path. This is opposite to the notation in Thornton and Marion, but I've used it because my θ is the usual polar angle in spherical coordinates.) To get this, consider the particle at its point of closest approach to the target (we're now considering the proper case where there is a force between particle and target). If we draw a line C from the target to the particle at that point, C will bisect the particle's path into two symmetric halves. Let the spherical angle of the particle at this point be θ_c. Since the particle's incident path is parallel to the z axis, the angle between the particle's incident path and its point of closest approach is also θ_c. Because C bisects the particle's path into symmetric halves, the angle between C and its asymptotic outgoing path is also θ_c. Therefore, the scattering angle Θ must be

$$\Theta = \pi - 2\theta_c \tag{14}$$

To visualize this, consider first the case where the two charges have the same sign. Then the particle is deflected away from the target, so $\theta_c < \frac{\pi}{2}$ and $\Theta > 0$. If the charges have opposite signs, the particle is deflected towards the target and $\theta_c > \frac{\pi}{2}$ and $\Theta < 0$.

The point of closest approach is found from 5 with the condition $\dot{r} = 0$ so for like charges

$$r_{min} = \frac{q_1 q_1 + \sqrt{(q_1 q_1)^2 + (8\pi\epsilon_0 E b)^2 m}}{8\pi\epsilon_0 E} \tag{15}$$

We can find the angle of closest approach by integrating from r_{min} to ∞:

$$\theta_c = b\int_{r_{min}}^{\infty} \frac{dr}{r\sqrt{r^2 - \frac{q_1 q_2}{4\pi\epsilon_0 E}r - b^2}} \tag{16}$$

Maple produces a result containing logarithms, but we can use a substitution to get an answer in terms of trig functions.

$$u \equiv \frac{1}{r} \tag{17}$$

$$du = -\frac{dr}{r^2} = -u^2 dr \tag{18}$$

$$dr = -\frac{du}{u^2} \tag{19}$$

$$\theta_c = b\int_0^{1/r_{min}} \frac{du}{\sqrt{1 - \frac{q_1 q_2}{4\pi\epsilon_0 E}u - b^2u^2}} \tag{20}$$

$$= -\arcsin\left(\frac{q_1 q_2}{\sqrt{(8\pi\epsilon_0 Eb)^2 + (q_1 q_1)^2}}\right) + \frac{\pi}{2} \tag{21}$$

$$b = \frac{q_1 q_2}{8\pi\epsilon_0 E}\tan\theta_c \tag{22}$$

$$= \frac{q_1 q_2}{8\pi\epsilon_0 E}\cot\frac{\Theta}{2} \tag{23}$$

where we used 14 in the last line.

That gives the impact parameter and scattering angle of a single particle, but most scattering experiments use a beam of particles, so can we get an expression for the probability that a particle will be scattered with a given angle? This is usually analyzed in terms of solid angle. A solid angle is essentially the area on a unit sphere that is cut by a cone (of arbitrary cross section) with its vertex at the centre of the sphere. Since the area of a unit sphere is 4π, the maximum solid angle is 4π.

In the case of scattering, we can view the experiment as a beam of particles that cross a plane (perpendicular to the z axis) on their way to the target. Those particles that cross a differential patch of area $d\sigma$ on the plane will be scattered to some differential solid angle $d\Omega$. As the potential is spherically symmetric, all particles with a given impact parameter b form a cylindrical stream heading towards the target and will be scattered at the same angle away from the target, so the solid angle into which they are scattered forms a ring centred on the z axis. Thus particles with impact parameters within a range db at distance b are scattered into the solid angle ring lying with $d\Theta$ of scattering angle Θ.

Imagine 2-d polar coordinates lying in the plane crossed by the incoming particles. Those particles crossing the area in the ring of width db at radius b are passing through a cross sectional area of

$$d\sigma = 2\pi b\, db \tag{24}$$

They are deflected into the solid angle ring of width $d\Theta$ and radius $\sin\Theta$, so the solid angle of the ring is

$$d\Omega = 2\pi \sin\Theta d\Theta \tag{25}$$

These two differentials are proportional to each other, so we can write

$$\begin{aligned} d\sigma &= \sigma(\Theta)\, d\Omega & (26)\\ 2\pi b\, db &= \sigma(\Theta)\, 2\pi \sin\Theta d\Theta & (27)\\ \sigma(\Theta) &= \frac{b}{\sin\Theta}\left|\frac{db}{d\Theta}\right| & (28)\end{aligned}$$

(Griffiths calls this function $D(\theta)$.) We've used the absolute value of the derivative because the larger the impact parameter b, the smaller the scattering angle, so $db/d\Theta < 0$ and we want $\sigma > 0$. Plugging in 23 we get

$$\begin{aligned} \sigma(\Theta) &= \left(\frac{q_1 q_2}{8\pi\epsilon_0 E}\right)^2 \frac{1}{\cos^2\Theta - 2\cos\Theta + 1} & (29)\\ &= \left(\frac{q_1 q_2}{8\pi\epsilon_0 E}\right)^2 \frac{1}{(1-\cos\Theta)^2} & (30)\\ &= \left(\frac{q_1 q_2}{8\pi\epsilon_0 E}\right)^2 \frac{1}{4\sin^4\frac{\Theta}{2}} & (31)\\ &= \left(\frac{q_1 q_2}{16\pi\epsilon_0 E \sin^2\frac{\Theta}{2}}\right)^2 & (32)\end{aligned}$$

The total cross section σ_T is the integral of $\sigma(\Theta)$ over all solid angles and gives the effective area of the target. That is, how far away from the target do we need to go in order for scattering not to occur? In the case of the Coulomb force

$$\begin{aligned} \sigma_T &= \int \sigma(\Theta)\, d\Omega & (33)\\ &= \left(\frac{q_1 q_2}{16\pi\epsilon_0 E}\right)^2 2\pi \int_0^\pi \frac{\sin\Theta d\Theta}{\sin^4\frac{\Theta}{2}} & (34)\end{aligned}$$

The integral comes out to

$$\int \frac{\sin\Theta d\Theta}{\sin^4\frac{\Theta}{2}} = -\frac{2}{\sin^2\frac{\Theta}{2}} \tag{35}$$

which is -2 when $\Theta = \pi$ but blows up when $\Theta \to 0$, so the total cross section is infinite, which is what we'd expect for a force with an infinite range.

PINGBACKS

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