QUANTUM SCATTERING: SCATTERING AMPLITUDE AND DIFFERENTIAL CROSS SECTION

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 11.2.

In quantum mechanics, the study of scattering is conceptually similar to the classical study of the scattering of waves. In a typical scattering experiment, a stream of particles travels in from (essentially) infinity and encounters a target (represented by a potential). As a result, particles can scatter away from the target in some radial direction. The probability of a given particle scattering in a particular direction can be measured by placing detectors or particle counters at various positions around the target.

In three dimensions, we can represent the incoming particle by a plane wave and the scattered particle by a spherical wave. For the plane wave, the surface probability density of the wave is constant as the wave travels (since the wave doesn't spread out), so we can represent this by the free particle wave function Ae^{ikz}, where $k = \sqrt{2mE}/\hbar$. We're using only the e^{ikz} term (and not the e^{-ikz} term), since we're considering a particle travelling in the $+z$ direction only.

For the spherical wave, the surface probability density decreases according to the inverse square law (just as the radiation from a star falls off according to the same law), so the wave function itself must decrease as $1/r$. The probability of scattering can, in general, depend on both the polar and azimuthal angles θ and ϕ (where the polar z axis is the direction of the incoming particle), so the outgoing wave function has the form $Af(\theta, \phi) e^{ikr}/r$ where f is the *scattering amplitude*. The complete wave function is then

$$\psi(r, \theta, \phi) = A\left(e^{ikz} + f(\theta, \phi)\frac{e^{ikr}}{r}\right) \tag{1}$$

If the potential of the scattering target is independent of ϕ, as would be the case for a potential that depends only on r, then the scattering probability cannot depend on ϕ either. It *can* still depend on θ, of course, since the direction of the incoming particle breaks the symmetry of the polar angle, even if the target is spherically symmetric. [Think of throwing stones

at a large spherical boulder. The probability of the stone scattering in a particular direction depends only on θ.] In this case, we get

$$\psi\left(r,\theta\right) = A\left(e^{ikz} + f\left(\theta\right)\frac{e^{ikr}}{r}\right) \tag{2}$$

These wave functions are valid for large distances from the target. The actual behaviour of the system within the region where the potential has an appreciable influence can be quite complicated and depends on the precise form of the potential function.

Although all real scattering experiments are done in three dimensions, we can generalize the above to lower dimensions. We've already looked at one-dimensional scattering, in which case the scattered particle can either continue on in the same direction or scatter directly backwards, so we get

$$\psi\left(z\right) = \begin{cases} Ae^{ikz} + Be^{-ikz} & z \ll 0 \\ Ce^{ikz} & z \gg 0 \end{cases} \tag{3}$$

where we're assuming that the potential's region of influence is localized within some finite distance of $z = 0$.

For two dimensions, the outgoing wave function is a circular wave and the probability density falls off as $1/r$ so the outgoing wave function must fall off as $1/\sqrt{r}$, giving

$$\psi\left(z,\theta\right) = A\left(e^{ikz} + f\left(\theta\right)\frac{e^{ikr}}{\sqrt{r}}\right) \tag{4}$$

Finally, we can relate f to physically measureable quantities by considering the *differential cross section*. In classical scattering, the idea is that if we take a cross section of the incoming particle beam and divide this cross section up into little regions of area $d\sigma$, then a particle that crosses the area element $d\sigma$ will (always, classically) scatter into an element of solid angle $d\Omega$. Depending on the scattering potential and the angle of scattering θ, there is a relation between the size of $d\sigma$ and the size of $d\Omega$. Basically, if we consider a larger element of cross sectional area, then the element of solid angle into which these particles scatter is also larger. The proportionality factor $d\sigma/d\Omega$ can depend on where we choose the element $d\sigma$ or, conversely, which element of $d\Omega$ we choose to detect so, for a spherically symmetric potential, $d\sigma/d\Omega$ will in general depend on θ, the scattering angle. That is, using the (non-standard) symbol $D\left(\theta\right)$ for the scattering cross-section

$$D\left(\theta\right) \equiv \frac{d\sigma}{d\Omega} \tag{5}$$

[Griffiths is quite right that this is a terrible name; I can remember being confused myself when I first encountered it. It's actually the ratio of a differential cross-section $d\sigma$ to a differential solid angle $d\Omega$.]

In the quantum case, we still assume that a particle incident in $d\sigma$ will scatter into $d\Omega$. However, here we can only deal in the probability that a particle will cross the area $d\sigma$ as it comes in, and the probability that it will scatter into $d\Omega$ as it leaves. The scattering assumption is that these two probabilities are equal. If the incident particle is travelling at speed v in the $+z$ direction, then the probability that it is found in a thin slice of space of area $d\sigma$ in time interval dt is

$$dP_i = |\psi_{incident}|^2\, dV = |A|^2\, v\, dt\, d\sigma \tag{6}$$

[You might wonder if it's correct to say that the incident particle's velocity is precisely v; what about the uncertainty principle? However, since we're dealing with a plane wave, the particle's position is totally unknown, so its momentum and hence velocity can be precisely known.]

The outgoing particle, also travelling at speed v, has a probability of being found in a thin radial slice of volume $v\, dt \times r^2 d\Omega$. Here, $v\, dt$ is the thickness of the slice and $r^2 d\Omega$ is the surface area of the slice. This time, the probability is

$$dP_o = |\psi_{outgoing}|^2\, dV = \frac{|Af|^2}{r^2} v\, dt\, r^2 d\Omega \tag{7}$$

Equating $dP_i = dP_o$ gives

$$D\left(\theta\right) \equiv \frac{d\sigma}{d\Omega} = |f\left(\theta\right)|^2 \tag{8}$$

Thus the differential cross section is equivalent to the square modulus of the scattering amplitude. The scattering amplitude f is what can, in principle, be calculated from the theory, so it provides the link between theory and experiment.

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