PHASE SHIFT IN ONE-DIMENSIONAL SCATTERING

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 11.5.

In a one-dimensional scattering problem, if we have an incident wave coming in from the left and scattering off a potential that is non-zero in the region $-a \le x \le 0$ and infinite for $x > 0$, then the wave effectively hits a brick wall at $x = 0$ and is fully reflected back in the $-x$ direction. Since the total probability of finding the particle can't change (it won't partially or fully disappear), the amplitude of the incoming wave function must equal the amplitude of the reflected wave function heading in the $-x$ direction. However, interaction with the potential in the region $-a \le x \le 0$ could change the phase of the wave function, since the phase doesn't affect the amplitude. That is, the wave function in the region $x < -a$ has the form

$$\psi(x) = A\left(e^{ikx} - e^{i(2\delta - kx)}\right) \tag{1}$$

where 2δ is the total phase change due to the reflection through the potential (the 2 is there since the wave function undergoes one phase shift of δ as it travels towards the origin and another phase shift of δ as it travels back towards $-x$, so 2δ is the total phase shift). The quantity $k = \sqrt{2mE}/\hbar$ as usual.

To find δ we must, as usual, solve the Schrödinger equation in the region $-a \le x \le 0$ and apply boundary conditions.

Example. Suppose the potential is a half-finite square well of form

$$V = \begin{cases} 0 & x < -a \\ -V_0 & -a \le x \le 0 \\ \infty & x > 0 \end{cases} \tag{2}$$

where V_0 is a positive constant. Then the solution of the Schrödinger equation inside the well is

$$\psi(x) = Be^{ik'x} + Ce^{-ik'x} \tag{3}$$

$$k' \equiv \frac{\sqrt{2m(E + V_0)}}{\hbar} \tag{4}$$

Because of the infinite barrier at $x=0$, we must have $\psi(0)=0$ so $C=-B$ and

$$\begin{aligned} \psi(x) &= B\left(e^{ik'x}-e^{-ik'x}\right) && (5)\\ &= 2iB\sin k'x && (6) \end{aligned}$$

Since the potential is finite at $x=-a$, both ψ and ψ' must be continuous there. This gives, from 1 and 6

$$\begin{aligned} A\left(e^{-ika}-e^{i(2\delta+ka)}\right) &= -2iB\sin k'a && (7)\\ ikA\left(e^{-ika}+e^{i(2\delta+ka)}\right) &= 2ik'B\cos k'a && (8) \end{aligned}$$

Dividing the second equation by the first gives

$$ik\frac{e^{-ika}+e^{i(2\delta+ka)}}{e^{-ika}-e^{i(2\delta+ka)}}=-k'\cot k'a \qquad (9)$$

We can solve for $e^{2i\delta}$ by multiplying the LHS by e^{-ika}/e^{-ika}, so we get

$$\begin{aligned} e^{2i\delta} &= e^{-2ika}\frac{ik+k'\cot k'a}{k'\cot k'a-ik} && (10)\\ &= -e^{-2ika}\frac{k-ik'\cot k'a}{k+ik'\cot k'a} && (11) \end{aligned}$$

Plugging this into 1 we see that the reflected wave is

$$\begin{aligned} \psi_{ref}(x) &= -Ae^{i(2\delta-kx)} && (12)\\ &= Ae^{-2ika}\frac{k-ik'\cot k'a}{k+ik'\cot k'a}e^{-ikx} && (13) \end{aligned}$$

The amplitude of the reflected wave is

$$\begin{aligned} \left|\psi_{ref}\right|^2 &= |A|^2\left|\frac{k-ik'\cot k'a}{k+ik'\cot k'a}\right|^2 && (14)\\ &= |A|^2\frac{k^2+(-k'\cot k'a)^2}{k^2+(k'\cot k'a)^2} && (15)\\ &= |A|^2 && (16) \end{aligned}$$

So the reflected wave has the same amplitude as the incident wave, as required.

For a very deep well, so that $V_0\gg E$, $k'\to\infty$ so from 11

$$
\begin{aligned}
e^{2i\delta} &\rightarrow -e^{-2ika}\left[\frac{-ik'\cot k'a}{ik'\cot k'a}\right]=e^{-2ika} && (17)\\
\delta &\rightarrow -ka && (18)
\end{aligned}
$$

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