

PHASE SHIFT IN THE SPHERICAL DELTA FUNCTION SHELL

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 11.7.

We can apply 3-d partial wave analysis using phase shiftsto the problem of the spherical delta function shell. Restricting our attention to the $l = 0$ term, we found earlier that the wave function for points outside the sphere is

$$(1) \quad \psi_{ext} = \frac{A}{kr} \left[\sin kr + ka_0 e^{ikr} \right]$$

where

$$(2) \quad a_0 = -\frac{\beta e^{-ika} \sin^2 ka}{(\beta \sin ka + ka \cos ka - iak \sin ka) k}$$

$$(3) \quad k \equiv \frac{\sqrt{2mE}}{\hbar}$$

$$(4) \quad \beta \equiv \frac{2ma\alpha}{\hbar^2}$$

and a is the radius of the sphere, and α is the strength of the delta function in the potential: $V(r) = \alpha \delta(r - a)$.

By comparing this form of the wave function with the phase shift form, we found that

$$(5) \quad a_l = \frac{1}{k} e^{i\delta_l} \sin \delta_l$$

To find the phase shift from 2, we need to put ka_0 in modulus-argument form. We can grind through the calculations by multiplying 2 top and bottom by the complex conjugate of the denominator and then finding the real and imaginary parts. This is just rather tedious algebra, so I got Maple to do it for me, with the results:

$$(6) \quad \Re(ka_0) = -\frac{\beta \sin^2(ka) (ka + \beta \sin(ka) \cos(ka))}{(ka)^2 + \beta^2 + 2ka\beta \sin(ka) \cos(ka) - \beta^2 \cos^2(ka)}$$

$$(7) \quad \Im(ka_0) = \frac{\beta^2 \sin^4(ka)}{(ka)^2 + \beta^2 + 2ka\beta \sin(ka) \cos(ka) - \beta^2 \cos^2(ka)}$$

From this, we get

$$(8) \quad \delta_0 = \arctan\left(\frac{\Im ka_0}{\Re ka_0}\right)$$

$$(9) \quad = \arctan\left(-\frac{\beta \sin^2(ka)}{ka + \beta \sin(ka) \cos(ka)}\right)$$

$$(10) \quad = -\arctan\left(\frac{\beta \sin^2(ka)}{ka + \beta \sin(ka) \cos(ka)}\right)$$

For some reason, Griffiths wants to express the answer using cotangents, so using $\arctan x = \operatorname{arccot} \frac{1}{x}$, we have

$$(11) \quad \delta_0 = -\operatorname{arccot}\left(\frac{ka + \beta \sin(ka) \cos(ka)}{\beta \sin^2(ka)}\right)$$

$$(12) \quad = -\operatorname{arccot}\left(\cot(ka) + \frac{ka}{\beta \sin^2(ka)}\right)$$