PHASE SHIFT IN THE SPHERICAL DELTA FUNCTION SHELL

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 11.7.

We can apply 3-d partial wave analysis using phase shiftsto the problem of the spherical delta function shell. Restricting our attention to the $l=0$ term, we found earlier that the wave function for points outside the sphere is

$$\psi_{ext}=\frac{A}{kr}\left[\sin kr+ka_0e^{ikr}\right] \tag{1}$$

where

$$a_0=-\frac{\beta e^{-ika}\sin^2 ka}{\left(\beta\sin ka+ka\cos ka-iak\sin ka\right)k} \tag{2}$$

$$k\equiv\frac{\sqrt{2mE}}{\hbar} \tag{3}$$

$$\beta\equiv\frac{2ma\alpha}{\hbar^2} \tag{4}$$

and a is the radius of the sphere, and α is the strength of the delta function in the potential: $V(r)=\alpha\delta(r-a)$.

By comparing this form of the wave function with the phase shift form, we found that

$$a_l=\frac{1}{k}e^{i\delta_l}\sin\delta_l \tag{5}$$

To find the phase shift from 2, we need to put ka_0 in modulus-argument form. We can grind through the calculations by multiplying 2 top and bottom by the complex conjugate of the denominator and then finding the real and imaginary parts. This is just rather tedious algebra, so I got Maple to do it for me, with the results:

$$\Re(ka_0) = -\frac{\beta \sin^2(ka)(ka + \beta \sin(ka)\cos(ka))}{(ka)^2 + \beta^2 + 2ka\beta \sin(ka)\cos(ka) - \beta^2\cos^2(ka)} \tag{6}$$

$$\Im(ka_0) = \frac{\beta^2 \sin^4(ka)}{(ka)^2 + \beta^2 + 2ka\beta \sin(ka)\cos(ka) - \beta^2\cos^2(ka)} \tag{7}$$

From this, we get

$$\delta_0 = \arctan\left(\frac{\Im ka_0}{\Re ka_0}\right) \tag{8}$$

$$= \arctan\left(-\frac{\beta \sin^2(ka)}{ka + \beta \sin(ka)\cos(ka)}\right) \tag{9}$$

$$= -\arctan\left(\frac{\beta \sin^2(ka)}{ka + \beta \sin(ka)\cos(ka)}\right) \tag{10}$$

For some reason, Griffiths wants to express the answer using cotangents, so using $\arctan x = \operatorname{arccot}\frac{1}{x}$, we have

$$\delta_0 = -\operatorname{arccot}\left(\frac{ka + \beta \sin(ka)\cos(ka)}{\beta \sin^2(ka)}\right) \tag{11}$$

$$= -\operatorname{arccot}\left(\cot(ka) + \frac{ka}{\beta \sin^2(ka)}\right) \tag{12}$$