PHASE SHIFT IN THE SPHERICAL DELTA FUNCTION SHELL

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 11.7.

We can apply 3-d partial wave analysis using phase shiftsto the problem of the spherical delta function shell. Restricting our attention to the $l=0$ term, we found earlier that the wave function for points outside the sphere is

$$\psi_{ext}=\frac{A}{kr}\left[\sin kr+ka_{0}e^{ikr}\right] \tag{0.1}$$

where

$$a_{0}=-\frac{\beta e^{-ika}\sin^{2}ka}{\left(\beta\sin ka+ka\cos ka-iak\sin ka\right)k} \tag{0.2}$$

$$k\equiv\frac{\sqrt{2mE}}{\hbar} \tag{0.3}$$

$$\beta\equiv\frac{2ma\alpha}{\hbar^{2}} \tag{0.4}$$

and a is the radius of the sphere, and α is the strength of the delta function in the potential: $V(r)=\alpha\delta(r-a)$.

By comparing this form of the wave function with the phase shift form, we found that

$$a_{l}=\frac{1}{k}e^{i\delta_{l}}\sin\delta_{l} \tag{0.5}$$

To find the phase shift from 0.2, we need to put ka_0 in modulus-argument form. We can grind through the calculations by multiplying 0.2 top and bottom by the complex conjugate of the denominator and then finding the real and imaginary parts. This is just rather tedious algebra, so I got Maple to do it for me, with the results:

$$\Re(ka_0) = -\frac{\beta \sin^2(ka)(ka + \beta \sin(ka)\cos(ka))}{(ka)^2 + \beta^2 + 2ka\beta \sin(ka)\cos(ka) - \beta^2 \cos^2(ka)} \tag{0.6}$$

$$\Im(ka_0) = \frac{\beta^2 \sin^4(ka)}{(ka)^2 + \beta^2 + 2ka\beta \sin(ka)\cos(ka) - \beta^2 \cos^2(ka)} \tag{0.7}$$

From this, we get

$$\delta_0 = \arctan\left(\frac{\Im ka_0}{\Re ka_0}\right) \tag{0.8}$$

$$= \arctan\left(-\frac{\beta \sin^2(ka)}{ka + \beta \sin(ka)\cos(ka)}\right) \tag{0.9}$$

$$= -\arctan\left(\frac{\beta \sin^2(ka)}{ka + \beta \sin(ka)\cos(ka)}\right) \tag{0.10}$$

For some reason, Griffiths wants to express the answer using cotangents, so using $\arctan x = \operatorname{arccot}\frac{1}{x}$, we have

$$\delta_0 = -\operatorname{arccot}\left(\frac{ka + \beta \sin(ka)\cos(ka)}{\beta \sin^2(ka)}\right) \tag{0.11}$$

$$= -\operatorname{arccot}\left(\cot(ka) + \frac{ka}{\beta \sin^2(ka)}\right) \tag{0.12}$$